9 Mixed Exercise

1 a $\overrightarrow{AB} = \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$

$\therefore \mathbf{r} = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\3\\-1\end{pmatrix} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{j} - \mathbf{k})$

b $\mathbf{c} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$

$= \begin{pmatrix}1\\-1\\3\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\3\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\\frac{7}{3}\end{pmatrix}$

$= \mathbf{i} + \mathbf{j} + \frac{7}{3}\mathbf{k}$

2 $A(7,-1,2)$ and $B(-1,3,8)$

$\overrightarrow{AB} = \begin{pmatrix}-8\\4\\6\end{pmatrix} = -2\begin{pmatrix}4\\-2\\-3\end{pmatrix}$

$\therefore$ **A** vector equation is

$\mathbf{r} = \begin{pmatrix}7\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}4\\-2\\-3\end{pmatrix}$

$\therefore \frac{x-7}{4} = \frac{y+1}{-2} = \frac{z-2}{-3}$

3 $(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + \lambda(2\mathbf{j} + 3\mathbf{k})$

4 a $\frac{x-3}{2} = \frac{y+2}{1} = \frac{z-1}{-1}$

b $\frac{0-3}{2} = \frac{a+2}{1} = \frac{b-1}{-1}$

$\Rightarrow a = -\frac{7}{2},\ b = \frac{5}{2}$

5 Use the equation $\mathbf{r}=\begin{pmatrix}1\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}$

When $\lambda=2$, $\mathbf{r}=\begin{pmatrix}7\\4\\-5\end{pmatrix}$, so the point $(7,4,-5)$ lies on l

$\begin{pmatrix}9\\3\\-6\end{pmatrix}=3\begin{pmatrix}3\\1\\-2\end{pmatrix}$ so these two vectors are parallel.

So an alternative form of the equation is $\mathbf{r}=\begin{pmatrix}7\\4\\-5\end{pmatrix}+\lambda\begin{pmatrix}9\\3\\-6\end{pmatrix}$

6 a $\mathbf{r}=\begin{pmatrix}-2\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\4\end{pmatrix}$

b Set $\lambda=2$

7 a $\overrightarrow{AB}=\begin{pmatrix}-1\\4\\-3\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}2\\3\\3\end{pmatrix}$

$\therefore \mathbf{r}=\begin{pmatrix}2\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\-3\end{pmatrix}+\mu\begin{pmatrix}2\\3\\3\end{pmatrix}$

b $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\-3\end{pmatrix}+\mu\begin{pmatrix}2\\3\\3\end{pmatrix}$

$x-2=-\lambda+2\mu$ (1)

$y+1=4\lambda+3\mu$ (2)

$z-2=-3\lambda+3\mu$ (3)

$(3)-(2)$: $z-y-3=-7\lambda \Rightarrow \lambda=\dfrac{y-z+3}{7}$

$\Rightarrow \mu=\dfrac{3y+4z-5}{21}$

Into (1):

$x-2=-\dfrac{y+3-z}{7}+2\left(\dfrac{3y+4z-5}{21}\right)$

$\Rightarrow 21(x-2)=-3(y+3-z)+2(3y+4z-5)$

$\Rightarrow 21x-3y-11z=23$

8 Three points on the plane are $A(6,0,0)$, $B(8,-3,0)$ and $C(10,0,3)$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, \ \overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

$$\therefore \mathbf{r} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

9 a $\overrightarrow{ML} = \begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

$$\overrightarrow{MN} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

b $\dfrac{\overrightarrow{ML}.\overrightarrow{MN}}{\left|\overrightarrow{ML}\right|\left|\overrightarrow{MN}\right|} = \dfrac{27}{5\sqrt{2}\ 3\sqrt{2}} = \dfrac{9}{10}$

10 a $\mathbf{a} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ p \\ q \end{pmatrix}$

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

Equation of l:

$$\mathbf{r} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

b Since C lies on l,

$$\begin{pmatrix} 3 \\ p \\ q \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

$3 = 9 - 3\mu$

$3\mu = 6$

$\mu = 2$

So $p = -2 + 4\mu = 6$ and $q = 1 + 5\mu = 11$

10 c $\cos\theta = \dfrac{\mathbf{OC}.\ \mathbf{AB}}{|\mathbf{OC}||\mathbf{AB}|}$

$\mathbf{OC}.\ \mathbf{AB} = \begin{pmatrix} 3 \\ 6 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = -9 + 24 + 55 = 70$

$|\mathbf{OC}| = \sqrt{3^2 + 6^2 + 11^2} = \sqrt{166}$

$|\mathbf{AB}| = \sqrt{(-3)^2 + 4^2 + 5^2} = \sqrt{50}$

$\cos\theta = \dfrac{70}{\sqrt{166}\sqrt{50}}$

$\theta = 39.8°\ \ (1\ \text{d.p.})$

d If **OD** and **AB** are perpendicular, **d** . (**b** – a) = 0

Since **d** lies on **AB**, use $\mathbf{d} = \begin{pmatrix} 9 - 3\mu \\ -2 + 4\mu \\ 1 + 5\mu \end{pmatrix}$

$\begin{pmatrix} 9 - 3\mu \\ -2 + 4\mu \\ 1 + 5\mu \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = 0$

$-3(9 - 3\mu) + 4(-2 + 4\mu) + 5(1 + 5\mu) = 0$

$-27 + 9\mu - 8 + 16\mu + 5 + 25\mu = 0$

$50\mu = 30$

$\mu = \dfrac{3}{5}$

$\mathbf{d} = \begin{pmatrix} 9 - \frac{9}{5} \\ -2 + \frac{12}{5} \\ 1 + 3 \end{pmatrix} = \dfrac{36}{5}\mathbf{i} + \dfrac{2}{5}\mathbf{j} + 4\mathbf{k}$

11 a $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$

$\therefore\ \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \mu\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$

b Set $\mu = -3$

11 c Let $\mathbf{a}=\begin{pmatrix}4\\-2\\0\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$

$\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}4\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}=4+4+0=8$

$|\mathbf{a}|=\sqrt{4^2+(-2)^2}=\sqrt{20}$

$|\mathbf{b}|=\sqrt{1^2+(-2)^2+2^2}=3$

$\therefore \cos\theta=\dfrac{8}{3\sqrt{20}}$

$\Rightarrow \theta=53.4^o$ (1.d.p.)

d Let G be a general point on l_1

$\overrightarrow{\mathbf{CG}}=\begin{pmatrix}1+4\mu\\-2-2\mu\\2\end{pmatrix}$

When **C** is closest to l_1, $\overrightarrow{\mathbf{CG}}$ is perpendicular to l_1.

$\therefore \begin{pmatrix}1+4\mu\\-2-2\mu\\2\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\0\end{pmatrix}=0$

$\Rightarrow \mu=-\dfrac{2}{5}$

$\therefore \overrightarrow{\mathbf{CG}}=\begin{pmatrix}-\frac{3}{5}\\-\frac{6}{5}\\2\end{pmatrix}$

$\Rightarrow |\overrightarrow{\mathbf{CG}}|=\sqrt{\left(-\frac{3}{5}\right)^2+\left(-\frac{6}{5}\right)^2+2^2}=\dfrac{\sqrt{145}}{5}$

12 a Line l_1: $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

Line l_2: $\mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$

Using the direction vectors:

$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 4 - 2 - 2 = 0$

Since the scalar product is zero, the directions are perpendicular.

b At an intersection point: $\begin{pmatrix} 3+\lambda \\ 4-2\lambda \\ -5+2\lambda \end{pmatrix} = \begin{pmatrix} 9+4\mu \\ 1+\mu \\ -2-\mu \end{pmatrix}$

$3 + \lambda = 9 + 4\mu$ $\quad (\times 2)$

$4 - 2\lambda = 1 + \mu$

$6 + 2\lambda = 18 + 8\mu$

$4 - 2\lambda = 1 + \mu$

Adding: $10 = 19 + 9\mu$

$\Rightarrow 9\mu = -9$

$\Rightarrow \mu = -1$

$3 + \lambda = 9 - 4$

$\Rightarrow \lambda = 2$

Intersection point: $\begin{pmatrix} 3+\lambda \\ 4-2\lambda \\ -5+2\lambda \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$

Position vector of **A** is a = 5**i** – **k**.

c Position vector of **B**: $\mathbf{b} = 10\mathbf{j} - 11\mathbf{k} = \begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$

For l_1, to give zero as the x component, $\lambda = -3$.

$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$

So **B** lies on l_1.

For l_2, to give – 11 as the z component, $\mu = 9$.

$\mathbf{r} = \begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + 9\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 45 \\ 10 \\ -11 \end{pmatrix}$

So **B** does not lie on l_2.

So Only one of the submarines passes through **B**.

12 d $|\mathbf{AB}| = \sqrt{(0-5)^2 + (10-0)^2 + [-11-(-1)]^2}$

$= \sqrt{(-5)^2 + 10^2 + (-10)^2}$

$= \sqrt{225} = 15$

Since 1 unit represents 100 m, the distance **AB** is $15 \times 100 = 1500$ m $= 1.5$ km.

13 Let A and B be general points on l_1 and l_2 respectively.

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 6+t \\ -2-2t-s \\ t \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l_1

$$\therefore \begin{pmatrix} 6+t \\ -2-2t-s \\ t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$\Rightarrow -2-2t-s = 0$

$\overrightarrow{AB}$ is perpendicular to l_2

$$\begin{pmatrix} 6+t \\ -2-2t-s \\ t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$$

$\Rightarrow 10 + 6t + 2s = 0$

Solving these simultaneous equations:

$s = 4,\ \text{t} = -3$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2} = 4.24$ (3 s.f.)

14 Let A and B be general points on l_1 and l_2 respectively.

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 6+t-3s \\ -2+2t+s \\ -s \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l_1

$$\therefore \begin{pmatrix} 6+t-3s \\ -2+2t+s \\ -s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 0$$

$\Rightarrow 20+t-11s=0$

$\overrightarrow{AB}$ is perpendicular to l_2

$$\begin{pmatrix} 6+t-3s \\ -2+2t+s \\ -s \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0$$

$\Rightarrow 2+5t-s=0$

Solving these simultaneous equations:

$s = \frac{49}{27}$, $\text{t} = -\frac{1}{27}$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} \frac{14}{27} \\ -\frac{7}{27} \\ -\frac{49}{27} \end{pmatrix}$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{\left(\tfrac{14}{27}\right)^2 + \left(-\tfrac{7}{27}\right)^2 + \left(-\tfrac{49}{27}\right)^2} = \sqrt{\frac{2646}{729}} = \sqrt{\frac{294}{81}} = \frac{7\sqrt{6}}{9} = 1.91 \text{ (3 s.f.)}$$

15 a $\mathbf{r}=\begin{pmatrix}1\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-1\end{pmatrix}$ and $\mathbf{a}=\begin{pmatrix}4\\3\\1\end{pmatrix}$

When $\lambda=0$, $\mathbf{r}=\begin{pmatrix}1\\1\\-2\end{pmatrix}$

When $\lambda=1$, $\mathbf{r}=\begin{pmatrix}3\\1\\-3\end{pmatrix}$

So the position vectors $\begin{pmatrix}1\\1\\-2\end{pmatrix}$, $\begin{pmatrix}3\\1\\-3\end{pmatrix}$ and $\begin{pmatrix}4\\3\\1\end{pmatrix}$ all lie on the plane

Suppose the plane has equation $ax+by+cz=1$

Substituting each of the coordinates into this equation gives:

$a+b-2c=1$

$3a+b-3c=1$

$4a+3b+c=1$

Solving these equations simultaneously gives:

$a=-\frac{2}{15}, b=\frac{9}{15}, c=-\frac{4}{15}$

Therefore the equation of the plane is

$-\frac{2}{15}x+\frac{9}{15}y-\frac{4}{15}z=1$

or $2x-9y+4z=-15$

So the required equation is $\mathbf{r}.\begin{pmatrix}2\\-9\\4\end{pmatrix}=-15$

15 b $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

When $\lambda = 0$, $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$

So the position vectors $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ all lie on the plane

Suppose the plane has equation $ax + by + cz = 1$

Substituting each of the coordinates into this equation gives:

$a + 2b + 2c = 1$

$3a + 3b - c = 1$

$3a + 5b + c = 1$

Solving these equations simultaneously gives:

$a = 1,\ b = -\frac{1}{2}, c = \frac{1}{2}$

Therefore the equation of the plane is

$x - \frac{1}{2}y + \frac{1}{2}z = 1$

or $2x - y + z = 2$

So the required equation is $\mathbf{r}.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2$

15 c $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 7 \\ 8 \\ 6 \end{pmatrix}$

When $\lambda = 0$, $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$

So the position vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 8 \\ 6 \end{pmatrix}$ all lie on the plane

Suppose the plane has equation $ax + by + cz = 1$

Substituting each of the coordinates into this equation gives:

$2a - b + c = 1$

$3a + b + 3c = 1$

$7a + 8b + 6c = 1$

Solving these equations simultaneously gives:

$a = \frac{8}{22}, b = -\frac{5}{22}, c = \frac{1}{22}$

Therefore the equation of the plane is

$\frac{8}{22}x - \frac{5}{22}y + \frac{1}{22}z = 1$

or $8x - 5y + z = 22$

So the required equation is $\mathbf{r}.\begin{pmatrix} 8 \\ -5 \\ 1 \end{pmatrix} = 22$

16 The line is in the direction $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. This lies in the plane.
$(2, -4, 1)$ is a point on the line. This also lies in the plane, as does the point (1, 1, 1).

$\therefore \begin{pmatrix}2\\-4\\1\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-5\\0\end{pmatrix}$ is a direction in the plane.

First obtain the equation of the plane in the form, $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$, then convert to Cartesian form.

Let $\mathbf{n}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ be perpendicular to the plane

So $\begin{pmatrix}3\\1\\2\end{pmatrix}.\begin{pmatrix}\mathbf{a}\\\mathbf{b}\\\mathbf{c}\end{pmatrix}=0$ and $\begin{pmatrix}1\\-5\\0\end{pmatrix}.\begin{pmatrix}\mathbf{a}\\\mathbf{b}\\\mathbf{c}\end{pmatrix}=0$

Therefore $3a+b+2c=0$ and $a-5b=0$

Choosing $a=5$ gives $b=1$ and $c=-8$

Therefore a normal vector is given by $\begin{pmatrix}5\\1\\-8\end{pmatrix}$

$\therefore$ The equation of the plane is

$\mathbf{r}\cdot(5\mathbf{i}+\mathbf{j}-8\mathbf{k})=(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(5\mathbf{i}+\mathbf{j}-8\mathbf{k})$

i.e: $5x+y-8z=-2$

This is a Cartesian equation of the plane.

17 a $\overrightarrow{\mathbf{AB}} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\overrightarrow{\mathbf{AC}} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

Let $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be perpendicular to the plane

Then $\mathbf{n}$ is perpendicular to both $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

So $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$

Therefore $2a - 2b + c = 0$ and $a + b - 2c = 0$

Choosing $a = 1$ gives $b - 2c = -1$ and $-2b + c = -2$

Solving simultaneously gives $b = \frac{5}{3}$ and $c = \frac{4}{3}$

Therefore a normal vector is given by $\begin{pmatrix} 1 \\ \frac{5}{3} \\ \frac{4}{3} \end{pmatrix}$, or $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$

Therefore a unit vector normal to the plane is $\frac{1}{\sqrt{3^2 + 5^2 + 4^2}}(3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

$= \frac{1}{\sqrt{50}}(3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

b The equation of the plane may be written as

$$\begin{aligned} \mathbf{r} \cdot (3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) &= (\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) \cdot (3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) \\ &= 3 + 15 + 12 \\ &= 30 \end{aligned}$$

i.e. $3x + 5y + 4z = 30$

c The perpendicular distance from the origin to the plane is

$$\frac{30}{\sqrt{3^2 + 5^2 + 4^2}} = \frac{30}{\sqrt{50}} = \frac{30\sqrt{50}}{50} = 3\sqrt{2}.$$

18 a The plane with vector equation
$\mathbf{r} = \mathbf{i} + s\mathbf{j} + t(\mathbf{i} - \mathbf{k})$
is perpendicular to $\mathbf{i} + \mathbf{k}$, as $(\mathbf{i} + \mathbf{k}) \cdot \mathbf{j} = 0$ and $(\mathbf{i} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{k}) = 1 - 1 = 0$
The plane also has equation
$\mathbf{r} \cdot (\mathbf{i} + \mathbf{k}) - \mathbf{i} \cdot (\mathbf{i} + \mathbf{k})$, as $\mathbf{i}$ is the position vector of a point on the plane.
i.e. $\mathbf{r} \cdot (\mathbf{i} + \mathbf{k}) = 1$

b The perpendicular distance from the origin to this plane is $\frac{1}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ or 0.707 (3 s.f.)

c The Cartesian form of the equation of the plane is $x + z = 1$.

19 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= 4\mathbf{i} - 3\mathbf{j}$
$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

Let $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be perpendicular to the plane

So $\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$ and $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$
Therefore $4a - 3b = 0$ and $2a + b + 5c = 0$

Choosing $a = 3$ gives $b = 4$ and $c = -2$

Therefore a normal vector is given by $\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$

Or any multiple of $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

b An equation of the plane containing **A**, **B** and **C** is
$\mathbf{r} \cdot (3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$
i.e. $3x + 4y - 2z - 5 = 0$

20 a $(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+3\mathbf{k})=2-10+9$
$=1$

$\therefore$ (2, –2, 3) lies on the plane Π_2

b $(2\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+3\mathbf{k})=2-5+3$
$=0$

$\therefore$ the normal to plane Π_1 is perpendicular to the normal to plane Π_2.
$\therefore$ Π_1 is perpendicular to Π_2.

c $\mathbf{r}=2\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{j}+\mathbf{k})$

d This line meets the plane Π_1 when
$\left[(2+2\lambda)\mathbf{i}+(-2-\lambda)\mathbf{j}+(3+\lambda)\mathbf{k}\right]\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=0$
i.e. $4+4\lambda+2+\lambda+3+\lambda=0$
i.e. $6\lambda+9=0$
$\therefore \lambda=-\frac{3}{2}$

Substitute $\therefore \lambda=-\frac{3}{2}$ into the equation of the line: then $\mathbf{r}=-\mathbf{i}-\frac{1}{2}\mathbf{j}+\frac{3}{2}\mathbf{k}$

i.e. The line meets Π_1 at the point $\left(-1,-\frac{1}{2},\frac{3}{2}\right)$

e The distance required is

$$\sqrt{(2-(-1))^2+\left(-2-\left(-\frac{1}{2}\right)\right)^2+\left(3-\frac{3}{2}\right)^2}=\sqrt{9+2\frac{1}{4}+2\frac{1}{4}}=\sqrt{13\frac{1}{2}}$$
$$=3.67\text{ (3 s.f.)}$$

21 a $l_1:\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}1+2\lambda\\1+\lambda\\-2\lambda\end{pmatrix}$ and $l_2:\mathbf{r}=\begin{pmatrix}1\\4\\-4\end{pmatrix}+\mu\begin{pmatrix}-3\\0\\1\end{pmatrix}=\begin{pmatrix}1-3\mu\\4\\-4+\mu\end{pmatrix}$

If l_1 and l_2 intersect, then $\begin{pmatrix}1+2\lambda\\1+\lambda\\-2\lambda\end{pmatrix}=\begin{pmatrix}1-3\mu\\4\\-4+\mu\end{pmatrix}$

$1+2\lambda=1-3\mu$ **(1)**
$1+\lambda=4$ **(2)**

Equation **(2)** gives $\lambda=3$
Substituting into **(1)** gives $7=1-3\mu$, so $\mu=-2$
Check for consistency: $-2\lambda=-6$ and $-4+\mu=-6$
$2+4\lambda=-5+2\mu$, so these equations are consistent.
Therefore l_1 and l_2 intersect.

21 b Substitute $\lambda = 3$ into line l_1, so $\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + 3\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}7\\4\\-6\end{pmatrix}$

Therefore the position vector of their point of intersection is $\begin{pmatrix}7\\4\\-6\end{pmatrix}$

c The cosine of the acute angle θ between the lines is the cosine of the acute angle between their respective direction vectors.

$$\begin{pmatrix}2\\1\\-2\end{pmatrix} . \begin{pmatrix}-3\\0\\1\end{pmatrix} = (2\times-3)+(1\times0)+(-2\times1) = -8$$

$$\left|\begin{pmatrix}2\\1\\-2\end{pmatrix}\right| = \sqrt{2^2+1^2+(-2)^2} = \sqrt{9} = 3 \text{ and } \left|\begin{pmatrix}-3\\0\\1\end{pmatrix}\right| = \sqrt{(-3)^2+0^2+1^2} = \sqrt{10}$$

So $\cos\theta = \left|\frac{-8}{3\sqrt{10}}\right| = \frac{8}{3\sqrt{10}}$

Therefore $\cos\theta = \frac{8}{3\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{8\sqrt{10}}{30} = \frac{4\sqrt{10}}{15}$, as required.

22 a $\begin{pmatrix}6\\8\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}3\\a\\2\end{pmatrix}$

$\lambda=-3 \Rightarrow a=11$

$\begin{pmatrix}6\\8\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}8\\6\\b\end{pmatrix}$

$\lambda=2 \Rightarrow b=7$

b $\left(\begin{pmatrix}6\\8\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$

$\Rightarrow\begin{pmatrix}6\\8\\5\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$

$\Rightarrow 6-8+5+\lambda(1+1+1)=0$

$\Rightarrow \lambda=-1$

$\therefore$ Position vector of **P** is $\begin{pmatrix}6\\8\\5\end{pmatrix}-\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}5\\9\\4\end{pmatrix}$

$\therefore \mathbf{P}(5,9,4)$

c $\sqrt{5^2+9^2+4^2}=\sqrt{122}$

23 a $\overrightarrow{\mathbf{AB}}=\begin{pmatrix}5\\2\\6\end{pmatrix}-\begin{pmatrix}6\\3\\4\end{pmatrix}=\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

b $\mathbf{r}=\begin{pmatrix}6\\3\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

23 c $\overrightarrow{\mathbf{CP}} = \begin{pmatrix} 2-\lambda \\ -7-\lambda \\ 2+2\lambda \end{pmatrix}$

$\begin{pmatrix} 2-\lambda \\ -7-\lambda \\ 2+2\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = 6\lambda + 9 = 0$

$\Rightarrow \lambda = -\frac{3}{2}$

$\therefore$ The position vector of $\mathbf{P}$ is

$\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ \frac{9}{2} \\ 1 \end{pmatrix}$

$\therefore \mathbf{P}\left(\frac{15}{2}, \frac{9}{2}, 1\right)$ or $\mathbf{P}(7.5, 4.5, 1)$

24 a $\begin{pmatrix} 3+2\lambda \\ -2+\lambda \\ 4-\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu \\ 12-2\mu \\ 8-\mu \end{pmatrix}$

$\mathbf{j}$ and $\mathbf{k}$ components $\Rightarrow \lambda = 2, \ \mu = 6$

Check $\mathbf{i}$ component: $3 + 2(2) = 7 = 1 + 6$

So the equations are consistent

Therefore the lines meet, and they meet at

$\begin{pmatrix} 3+2(2) \\ -2+2 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} \qquad \therefore A(7,0,2)$

b Let $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 2 - 2 + 1 = 1$

$|\mathbf{a}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

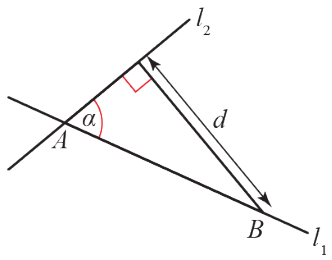
$|\mathbf{b}| = \sqrt{6}$

$\therefore \cos\theta = \frac{1}{6}$

$\Rightarrow \theta = 80.4^o$ (1 d.p.)

c Set $\lambda = 2$

24 d The shortest distance of $\mathbf{B}$ to the line l_2 is given by $\mathbf{d} = \left|\overrightarrow{\mathbf{BA}}\right| \sin\alpha$



$\overrightarrow{\mathbf{OA}} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$ and $\overrightarrow{\mathbf{OB}} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$

$\overrightarrow{\mathbf{BA}} = \overrightarrow{\mathbf{OA}} - \overrightarrow{\mathbf{OB}} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\left|\overrightarrow{\mathbf{BA}}\right| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

So $\mathbf{d} = \sqrt{6}\sin 80.4° = 2.42$

25 a $\overrightarrow{AP} = \mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$

$\mathbf{n} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

$\mathbf{a} \cdot \mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = -3$

$|\mathbf{a}| = \sqrt{3^2 + (-3)^2} = \sqrt{18}$

$|\mathbf{n}| = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17}$

$\therefore \sin\theta = \frac{|-3|}{\sqrt{18}\sqrt{17}} = \frac{3}{\sqrt{18}\sqrt{17}}$

$\Rightarrow \theta = 10^o$ to the nearest degree

b $\frac{|1(2) + 2(-2) + 2(3) - 1|}{\sqrt{2^2 + (-2)^2 + 3^2}} = \frac{3}{\sqrt{17}} = \frac{3\sqrt{17}}{17}$

26 a Let l_1 denote the path of the first aeroplane, and l_2 the second.

$$l_1:\ \mathbf{r}=\begin{pmatrix}120\\-80\\13\end{pmatrix}+\lambda\begin{pmatrix}80\\100\\-8\end{pmatrix}$$

$$l_2:\ \mathbf{r}=\begin{pmatrix}-20\\35\\5\end{pmatrix}+\mu\begin{pmatrix}10\\-2\\0.1\end{pmatrix}$$

$$\begin{pmatrix}120+80\lambda\\-80+100\lambda\\13-8\lambda\end{pmatrix}=\begin{pmatrix}-20+10\mu\\35-2\mu\\5+0.1\mu\end{pmatrix}$$

i and **k** components $\Rightarrow \lambda=\frac{3}{4},\ \mu=20$

Check **j** component: $-80+100\left(\frac{3}{4}\right)=-5=35-2(20)$

So the equations are consistent

Therefore the paths intersect, and they intersect at

$$\begin{pmatrix}120\\-80\\13\end{pmatrix}+\frac{3}{4}\begin{pmatrix}80\\100\\-8\end{pmatrix}=\begin{pmatrix}180\\-5\\7\end{pmatrix}\Rightarrow(180,-5,7)$$

b The planes pass through the same point, but not necessarily at the same time.

Challenge

1 **a** $-2(-2x+y-3z)=(-2)\times(-5)$ gives $4x-2y+6z=10$

b matrix **A** is singular if det **A** = 0

$4(c+3b)+2(-2c+3a)+6(-2b-a)=6a-6a+12b-12b+4c-4c=0$

c **i** $a=2n$, $b=-n$, $c=30$ where $n\in\mathbb{R}$, $n\neq 3$

ii $a=6$, $b=-3$ and $c=9$

Challenge

2 Coordinates $A(4,-4,5), B(0,4,1)$ and $C(0,0,5)$ lie on the circumference of the circle.

Let O be the centre of the circle.

Then A, B, C and O lie on the same plane.

Without loss of generality, suppose the plane has equation $ax+by+cz=1$

Then substituting point A gives: $4a-4b+5c=1$ **(1)**

substituting point B gives: $4b+c=1$ **(2)**

substituting point C gives: $5c=1$ **(3)**

From equation **(3)**, $c=\frac{1}{5}$

Substituting into **(2)** gives $4b+\frac{1}{5}=1$

$4b=\frac{4}{5}$

$b=\frac{1}{5}$

Substituting b and c in equation **(1)** gives $4a-\frac{4}{5}+\frac{5}{5}=1$

$4a=\frac{4}{5}$

$a=\frac{1}{5}$

So the equation of the plane is $\frac{x}{5}+\frac{y}{5}+\frac{z}{5}=1$

Or $x+y+z=5$ **(4)**

The centre of the circle lies on the intersection of any two perpendicular bisectors chosen from AB, BC or AC.

Each perpendicular bisector must also lie on the plane $x+y+z=5$

To find the equation of the perpendicular bisector of **AB**:

The mid-point of **AB** is $\mathbf{M}\left(\frac{4+0}{2},\frac{-4+4}{2},\frac{5+1}{2}\right)=\mathbf{M}(2,0,3)$

Therefore the equation of the line **MO** may be written as $\mathbf{r}=\begin{pmatrix}2\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$

$\overrightarrow{\mathbf{AB}}=\begin{pmatrix}-4\\8\\4\end{pmatrix}$ and $\overrightarrow{\mathbf{AB}}$ is perpendicular to the direction of **MO**, so $\overrightarrow{\mathbf{AB}}\cdot\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=0$

i.e. $\begin{pmatrix}-4\\8\\4\end{pmatrix}\cdot\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=0$, so $-4\alpha+8\beta-4\gamma=0$ **(5)**

The line $\mathbf{r}=\begin{pmatrix}2\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$ lies on the plane $x+y+z=5$ for all values of λ.

Choosing $\lambda=1$, you have $\mathbf{r}=\begin{pmatrix}2+\alpha\\\beta\\3+\gamma\end{pmatrix}$

Challenge
2 continued

Solving $\mathbf{r}=\begin{pmatrix}2+\alpha\\ \beta\\ 3+\gamma\end{pmatrix}$ and $x+y+z=5$ simultaneously gives $(2+\alpha)+\beta+(3+\gamma)=5$, or

$\alpha+\beta+\gamma=0$ **(6)**

Equation **(6)** is equivalent to $4\alpha+4\beta+4\gamma=0$

Adding this to equation **(5)** gives $12\beta=0$, so $\beta=0$ and $\alpha=-\gamma$

Therefore $\mathbf{r}=\begin{pmatrix}2+\alpha\\ \beta\\ 3+\gamma\end{pmatrix}=\begin{pmatrix}2+\alpha\\ 0\\ 3-\alpha\end{pmatrix}$

Now to find the equation of the perpendicular bisector of **BC**:

The mid-point of **BC** is $\mathbf{M}_1\left(\frac{0+0}{2},\frac{4+0}{2},\frac{1+5}{2}\right)=\mathbf{M}_1(0,2,3)$

Therefore the equation of the line $\mathbf{M}_1\mathbf{O}$ may be written as $\mathbf{r}=\begin{pmatrix}0\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix}$

$\overrightarrow{\mathbf{BC}}=\begin{pmatrix}0\\-4\\4\end{pmatrix}$ and $\overrightarrow{\mathbf{BC}}$ is perpendicular to the direction of $\mathbf{M}_1\mathbf{O}$, so $\overrightarrow{\mathbf{BC}}\cdot\begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix}=0$

i.e. $\begin{pmatrix}0\\-4\\4\end{pmatrix}\cdot\begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix}=0$, so $-4\beta_1+4\gamma_1=0$, or $\beta_1=\gamma_1$ **(7)**

The line $\mathbf{r}=\begin{pmatrix}0\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix}$ lies on the plane $x+y+z=5$ for all values of λ.

Choosing $\lambda=1$, you have $\mathbf{r}=\begin{pmatrix}\alpha_1\\ 2+\beta_1\\ 3+\gamma_1\end{pmatrix}$

Solving $\mathbf{r}=\begin{pmatrix}\alpha_1\\ 2+\beta_1\\ 3+\gamma_1\end{pmatrix}$ and $x+y+z=5$ simultaneously gives $\alpha_1+(2+\beta_1)+(3+\gamma_1)=5$, or

$\alpha_1+\beta_1+\gamma_1=0$ **(8)**

Solving equations **(7)** and **(8)** simultaneously gives $\alpha_1=-\beta_1-\gamma_1=-\gamma_1-\gamma_1=-2\gamma_1$

Therefore $\mathbf{r}=\begin{pmatrix}\alpha_1\\ 2+\beta_1\\ 3+\gamma_1\end{pmatrix}=\begin{pmatrix}-2\gamma_1\\ 2+\gamma_1\\ 3+\gamma_1\end{pmatrix}$

Challenge
2 continued

Now consider the equations of the two perpendicular bisectors:

$\mathbf{r} = \begin{pmatrix} 2+\alpha \\ 0 \\ 3-\alpha \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2\gamma_1 \\ 2+\gamma_1 \\ 3+\gamma_1 \end{pmatrix}$

These meet at the centre of the circle, O.

Equating the $\mathbf{j}$ components gives $0 = 2+\gamma_1$, so $\gamma_1 = -2$

Therefore the two bisectors must meet at the position vector $\mathbf{r} = \begin{pmatrix} -2(-2) \\ 2+(-2) \\ 3+(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

The centre of the circle is therefore $O(4,0,1)$

The radius is the distance between O and any of the points A, B or C

Consider the point $C(0,0,5)$

Then $OC = \sqrt{(0-4)^2 + (0-0)^2 + (5-1)^2} = \sqrt{32} = 4\sqrt{2}$

Therefore the circle has radius $4\sqrt{2}$