Exercise 9F

1 Let A be a general point on the first line and B be a general point on the second line,

then $\overrightarrow{AB} = \begin{pmatrix} -2 \\ +2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$, where $t = \mu - \lambda$.

Find the minimum value of the quadratic by using calculus, or completion of the square.

Let the distance $AB = x$

then $x^2 = (-2-3t)^2 + (2-4t)^2 + (5t)^2 \;= 8 - 4t + 50t^2$

The minimum value of x^2 occurs when $t = \frac{1}{25}$

So $x^2 = 8 - \frac{4}{25} + \frac{50}{625}$

$= \frac{198}{25}$

$\therefore x = \frac{\sqrt{198}}{5}$ or 2.81 (3 s.f.)

2 Let the lines respectively be l_1 and l_2

Let A and B be general points on l_1 and l_2 respectively.

$\therefore \overrightarrow{AB} = \begin{pmatrix} 5+\lambda+4\mu \\ -3-\lambda-2\mu \\ 1+\lambda+3\mu \end{pmatrix}$

$\overrightarrow{AB}$ perpendicular to l_1

$\therefore \begin{pmatrix} 5+\lambda+4\mu \\ -3-\lambda-2\mu \\ 1+\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 0$

$\Rightarrow \lambda + 3\mu = -3$

$\overrightarrow{AB}$ perpendicular to l_2

$\begin{pmatrix} 5+\lambda+4\mu \\ -3-\lambda-2\mu \\ 1+\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = 0$

$\Rightarrow 9\lambda + 29\mu = -29$

Solving these simultaneous equations,

$\lambda = 0,\ \mu = -1$

$\therefore \overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$

3 a $\begin{pmatrix}7+6\lambda\\3+2\lambda\\1-4\lambda\end{pmatrix}=\begin{pmatrix}-1\\1-2\mu\\2\end{pmatrix}$

i component $\Rightarrow\ \lambda=-\frac{4}{3}$

k component $\Rightarrow\ \lambda=-\frac{1}{4}$

So the equations aren't consistent

Therefore, l_1 and l_2 do not intersect

Let A and B be general points on l_1 and l_2 respectively.

$\therefore\overrightarrow{AB}=\begin{pmatrix}-8-6\lambda\\-2-2\lambda-2\mu\\1+4\lambda\end{pmatrix}$

$\overrightarrow{AB}$ is perpendicular to l_1

$\therefore\begin{pmatrix}-8-6\lambda\\-2-2\lambda-2\mu\\1+4\lambda\end{pmatrix}\cdot\begin{pmatrix}6\\2\\-4\end{pmatrix}=0$

$\Rightarrow 14\lambda+\mu=-14$

$\overrightarrow{AB}$ is perpendicular to l_2

$\begin{pmatrix}-8-6\lambda\\-2-2\lambda-2\mu\\1+4\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\-2\\0\end{pmatrix}=0$

$\Rightarrow\lambda+\mu=-1$

Solving these simultaneous equations,

$\lambda=-1,\ \mu=0$

$\therefore\overrightarrow{AB}=\begin{pmatrix}-2\\0\\-3\end{pmatrix}$

$\Rightarrow|\overrightarrow{AB}|=\sqrt{(-2)^2+(-3)^2}=\sqrt{13}=3.61\text{ (3 s.f.)}$

3 b Assume that l_1 and l_2 meet:

$$\begin{pmatrix} 2+2\lambda \\ 1-2\lambda \\ -2+2\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu \\ -1-\mu \\ 3+\mu \end{pmatrix}$$

i.e. $2+2\lambda = 1+\mu$ **(1)**

$1-2\lambda = -1-\mu$ **(2)**

$-2+2\lambda = 3+\mu$ **(3)**

Adding equations **(1)** and **(2)** gives 3 = 0

This is a contradiction.

$\therefore$ Lines do not meet.

The lines are in fact parallel as $2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is a multiple of $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

The distance between them is found by considering A on line l_1 and B on line l_2.

Then $\overrightarrow{AB} = -\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\left|\overrightarrow{AB}\right|^2 = x^2 = (-1+t)^2 + (-2-t)^2 + (5+t)^2$$

$$= 1 - 2t + t^2 + 4 + 4t + t^2 + 25 + 10t + t^2$$

$$= 30 + 12t + 3t^2$$

The minimum value of x^2 occurs when $\frac{\mathrm{d}(x^2)}{\mathrm{d}t} = 0$

$$\frac{\mathrm{d}(x)^2}{\mathrm{d}t} = 12 + 6t$$

When $\frac{\mathrm{d}(x)^2}{\mathrm{d}t} = 0, t = -2$

$$\therefore x^2 = 30 - 24 + 12$$

$$= 18$$

$$\therefore x = \sqrt{18} = 3\sqrt{2} \text{ or } 4.24 \text{ (3 s.f.)}$$

3 c Assume that l_1 and l_2 meet. Then

$$\begin{pmatrix} 1+2\lambda \\ 1+\lambda \\ 5-2\lambda \end{pmatrix} = \begin{pmatrix} -1+\mu \\ -1+\mu \\ 2+\mu \end{pmatrix} \quad \begin{matrix} \textbf{(1)} \\ \textbf{(2)} \\ \textbf{(3)} \end{matrix}$$

i.e. $1+2\lambda = -1+\mu$ **(1)**

$1+\lambda = -1+\mu$ **(2)**

$5-2\lambda = 2+\mu$ **(3)**

Comparing equations **(1)** and **(2)** gives $1+2\lambda = 1+\lambda$, so $\lambda = 0$

Substituting $\lambda = 0$ in **(1)** gives $\mu = 2$

Substituting these values in (3) gives $5-2\times 0 = 2+2$, which is a contradiction

$\therefore$ Lines do not meet.

Let A and B be general points on l_1 and l_2 respectively.

$$\therefore \overrightarrow{AB} = \begin{pmatrix} -2+\mu-2\lambda \\ -2+\mu-\lambda \\ -3+\mu+2\lambda \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l_1

$$\therefore \begin{pmatrix} -2+\mu-2\lambda \\ -2+\mu-\lambda \\ -3+\mu+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0$$

$\Rightarrow \mu - 9\lambda = 0$

$\overrightarrow{AB}$ is perpendicular to l_2

$$\begin{pmatrix} -2+\mu-2\lambda \\ -2+\mu-\lambda \\ -3+\mu+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$\Rightarrow -7+3\mu-\lambda = 0$

Solving these simultaneous equations,

$$\lambda = \frac{7}{26}, \ \mu = \frac{63}{26}$$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} -\frac{3}{26} \\ \frac{4}{26} \\ -\frac{1}{26} \end{pmatrix}$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{\left(-\tfrac{3}{26}\right)^2 + \left(\tfrac{4}{26}\right)^2 + \left(-\tfrac{1}{26}\right)^2} = \frac{1}{\sqrt{26}} = 0.196 \ (3 \ s.f.)$$

4 Let A be the point (4,1, –1) and B be the point (3 + 2t, –1 –t, 2 –t) which lies on the line.

Find the distance between (4, 1, –1) and (3 + 2t, –1 – t, 2 –t) at a point on the line.

Then $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$

$$= [4-(3+2t), 1-(-1-t), -1-(2-t)]$$
$$= [1-2t, 2+t, -3+t]$$
$$\therefore \left|\overrightarrow{BA}\right|^2 = (1-2t)^2 + (2+t)^2 + (-3+t)^2$$
$$= 6t^2 - 6t + 14$$

$\left|\overrightarrow{BA}\right|$ is a minimum when $\left|\overrightarrow{BA}\right|^2$ is minimum

This minimum value can be found by calculus or completion of the square.

$$\left|\overrightarrow{BA}\right|^2 = 6(t^2 - t) + 14$$
$$= 6\left(t - \frac{1}{2}\right)^2 + 14 - \frac{6}{4}$$

This is a minimum when $t = \frac{1}{2}$ and

$$\left|\overrightarrow{BA}\right|^2 = 14 - 1\frac{1}{2} = 12\frac{1}{2}$$
$$\therefore \left|\overrightarrow{BA}\right| = \sqrt{12\frac{1}{2}} = 3.54 \text{ (3 s.f.)}$$

5 a The distance from the origin to the plane $\mathbf{r}\cdot(6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 55$ is $\frac{55}{\sqrt{6^2 + 6^2 + (-7)^2}}$

$$= \frac{55}{\sqrt{121}}$$
$$= \frac{55}{11}$$
$$= 5$$

First find the distance from the origin to each plane, then subtract.

The distance from the origin to the plane $\mathbf{r}\cdot(6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 22$ is $\frac{22}{\sqrt{6^2 + 6^2 + (-7)^2}}$

$$= \frac{22}{11}$$
$$= 2$$

$\therefore$ The distance between the planes is $5 - 2 = 3$

5 b Let **n** = (n_1, n_2, n_3),
Take the scalar product of **n** with each of the two direction vectors of either of the planes
(The first plane has been used in this solution)

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = 4n_1 + n_3 = 0 \qquad \textbf{(1)}$$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 3 \\ 3 \end{pmatrix} = 8n_1 + 3n_2 + 3n_3 = 0 \qquad \textbf{(2)}$$

Since if two vectors are perpendicular, then their scalar product is zero.

Let $n_1 = k$
Substitute $n_1 = k$ into **(1)** to find $n_3 = -4k$
Substitute $n_1 = k$ and $n_3 = -4k$ into **(2)** to find $n_2 = \frac{4}{3}k$

So $\mathbf{n} = \left(k,\ \frac{4}{3}k,\ -4k \right)$
Let $k = 3$
(You could choose any value for k other than zero, 3 is convenient as it removes the fraction.)
$\mathbf{n} = (3,\ 4,\ -12)$
So the Cartesian form of the first plane is
$3x + 4y - 12z = 13$
To find the distance between the planes choose a point on the second plane and use

$$\frac{|a\alpha + b\beta + c\gamma + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Choose (14, 2, 2) as the point on the second plane and substitute to find

$$\text{Distance between planes} = \frac{|3(14) + 4(2) + (-12)(2) + (-13)|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$

$$= \frac{42 + 8 - 24 - 13}{\sqrt{169}}$$

$$= \frac{13}{13}$$

$$= 1$$

So the distance between the planes is 1.

6 a The length of the normal vector $10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}$ is $\sqrt{10^2 + 10^2 + 23^2} = \sqrt{729} = 27$
$\therefore\ \frac{1}{27}(10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k})$ is a unit vector normal to the plane.
The plane has equation
$\mathbf{r} \cdot (10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}) = 81$
or $\mathbf{r} \cdot \frac{1}{27}(10\mathbf{i} + 10\mathbf{j} + 23\mathbf{k}) = \frac{81}{27} = 3$
$\therefore$ The perpendicular distance from the origin to the plane is 3.

6 b A plane parallel to π through the point (–1, –1, 4) has equation

$$\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=(-\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$=\frac{-10}{27}-\frac{10}{27}+\frac{92}{27}$$
$$=\frac{72}{27}$$
$$=\frac{8}{3}$$

∴ The perpendicular distance from the origin to this new plane is $2\frac{2}{3}$

This distance between the planes is $3-2\frac{2}{3}=\frac{1}{3}$

∴ The perpendicular distance from the point $(-1,-1,4)$ to the plane π is $\frac{1}{3}$

c A plane parallel to π through the point (2, 1, 3) has equation

$$\mathbf{r}\cdot\frac{1}{24}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=(2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$=\frac{20}{27}+\frac{10}{27}+\frac{69}{27}$$
$$=\frac{99}{27}$$
$$=\frac{11}{3}$$

∴ The perpendicular distance from the origin to this new plane is $3\frac{2}{3}$

∴ The distances between this plane and π is $3\frac{2}{3}-3=\frac{2}{3}$

∴ The perpendicular distance from $(2,1,3)$ to π is $\frac{2}{3}$

d A plane parallel to π through the point $(6,12,-9)$ has equation

$$\mathbf{r}\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})=(6\mathbf{i}+12\mathbf{j}-9\mathbf{k})\cdot\frac{1}{27}(10\mathbf{i}+10\mathbf{j}+23\mathbf{k})$$
$$=\frac{60}{27}+\frac{120}{27}-\frac{207}{27}$$
$$=-\frac{27}{27}$$
$$=-1$$

∴ The perpendicular distance from the origin to this new plane is 1, in the opposite direction.
∴ The distance between this plane and π is $3-(-1)=4$
∴ The perpendicular distance from $(2, 1, 3)$ to π is 4.

7 Let **M** be the midpoint of the line segment joining P to its reflection in l.
Then, for some λ

$$\overrightarrow{PM} = \begin{pmatrix} -1+\lambda \\ -1+2\lambda \\ 1-\lambda \end{pmatrix}$$

$\overrightarrow{PM}$ must be perpendicular to l

$$\therefore \begin{pmatrix} -1+\lambda \\ -1+2\lambda \\ 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$$

$$\Rightarrow \lambda = \frac{2}{3}$$

$$\therefore \overrightarrow{PM} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

The reflection of P in l has position vector

$$\overrightarrow{OP} + 2\overrightarrow{PM} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ \frac{2}{3} \\ \frac{8}{3} \end{pmatrix}$$

$$\therefore \left(\frac{7}{3}, \frac{2}{3}, \frac{8}{3}\right)$$

8 **a** $\dfrac{|1(-2)+0(1)+3(1)-5|}{\sqrt{(-2)^2+1^2+1^2}} = \dfrac{4}{\sqrt{6}} = \dfrac{2\sqrt{6}}{3}$

b A unit normal vector to the plane is $\dfrac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

The line segment along the shortest distance from P to the plane is normal to the plane,

and the distance is $\dfrac{2\sqrt{6}}{3}$

Therefore Q has position vector

$$\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + 2\left(\frac{2\sqrt{6}}{3}\right)\left(\frac{1}{\sqrt{6}}\right)\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} \\ \frac{4}{3} \\ \frac{13}{3} \end{pmatrix}$$

$$\therefore Q\left(-\frac{5}{3}, \frac{4}{3}, \frac{13}{3}\right)$$

9 a The kestrel's path in vector form is

$$\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -5 \\ 1.2 \end{pmatrix},\ 0 \leqslant \lambda \leqslant 1$$

Let **G** be the position of the birdwatcher.

Let **H** be a general point in the path. Then,

$$\overrightarrow{GH} = \begin{pmatrix} -2+9\lambda \\ 1-5\lambda \\ -0.7+1.2\lambda \end{pmatrix}$$

The shortest distance between the kestrel and the birdwatcher will occur when $\overrightarrow{GH}$ is perpendicular to the path.

$$\therefore \begin{pmatrix} -2+9\lambda \\ 1-5\lambda \\ -0.7+1.2\lambda \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -5 \\ 1.2 \end{pmatrix} = 0$$

$$\Rightarrow \lambda = \frac{35}{158}$$

$$\therefore \overrightarrow{GH} = \begin{pmatrix} -2+9\left(\frac{35}{158}\right) \\ 1-5\left(\frac{35}{158}\right) \\ -0.7+1.2\left(\frac{35}{158}\right) \end{pmatrix}$$

$$\therefore |\overrightarrow{GH}| = 0.45 \quad \text{(2 d.p.)}$$

$$\Rightarrow \quad |\overrightarrow{GH}| < 0.5$$

So the birdwatcher will be able to spot the kestrel.

b In practice, the bird is very unlikely to fly in a straight line from A to B

10 a $$\frac{|4(3)-(-2)+8(4)-6|}{\sqrt{3^2+(-2)^2+4^2}} = \frac{40}{\sqrt{29}} = \frac{40\sqrt{29}}{29}$$

b $$\left(\lambda\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

$$= \lambda\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

$$= \lambda(4-10+6) + \mu(-6+0+6) = 0$$

10 c $\mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$

$\mathbf{n}_1 \cdot \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} = 6 - 10 + 8 = 4$

$|\mathbf{n}_1| = \sqrt{29}, \quad |\mathbf{n}_2| = \sqrt{33}$

$\therefore \cos\theta = \dfrac{4}{\sqrt{29}\sqrt{33}}$

$\Rightarrow \theta = 82.6^o \quad \text{(1d.p.)}$

11 a A vector equation for l_1 is

$\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

Let M be a general point on l_1

$\overrightarrow{AM} = \begin{pmatrix} -5 + 2\lambda \\ 3 - \lambda \\ -3 - 2\lambda \end{pmatrix}$

$|\overrightarrow{AM}|$ is minimum when $\overrightarrow{AM}$ is perpendicular to l_1

$\therefore \begin{pmatrix} -5 + 2\lambda \\ 3 - \lambda \\ -3 - 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$

$\Rightarrow \lambda = \dfrac{7}{9}$

$\therefore \overrightarrow{AM} = \dfrac{1}{9} \begin{pmatrix} -31 \\ 20 \\ -41 \end{pmatrix}$

$\Rightarrow |\overrightarrow{AM}| = \dfrac{13\sqrt{2}}{3}$

b A vector equation is

$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -31 \\ 20 \\ -41 \end{pmatrix}$

So a cartesian equation is

$\dfrac{x-3}{-31} = \dfrac{y+1}{20} = \dfrac{z-2}{-41}$

12 First find the intersection between l_1 and the plane.

$$\begin{pmatrix} 2+4\lambda \\ -1-3\lambda \\ 2+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 1+3\lambda = 4$$

$\Rightarrow \lambda = 1$

$\therefore (6,-4,6)$

Choose any other point along l_1, and reflect

it in the plane to obtain e.g. $\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$

$$r = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$