Exercise 9D

1 a Direction vectors are $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -9 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}\,.\,\mathbf{b} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -9 \end{pmatrix} = 6 - 5 + 9 = 10$$

$$|\mathbf{a}| = \sqrt{3^2 + (-5)^2 + (-1)^2} = \sqrt{35}$$

$$|\mathbf{b}| = \sqrt{2^2 + 1^2 + (-9)^2} = \sqrt{86}$$

$$\cos\theta = \frac{10}{\sqrt{35}\sqrt{86}}$$

$\theta = 79.5°$ (1 d.p.)

The acute angle between the lines is 79.5° (1 d.p.)

b Direction vectors are $\mathbf{a} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}\,.\,\mathbf{b} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix} = 8 + 2 + 3 = 13$$

$$|\mathbf{a}| = \sqrt{(-2)^2 + (-1)^2 + 3^2} = \sqrt{14}$$

$$|\mathbf{b}| = \sqrt{(-4)^2 + (-2)^2 + 1^2} = \sqrt{21}$$

$$\cos\theta = \frac{13}{\sqrt{14}\sqrt{21}}$$

$\theta = 40.7°$ (1 d.p.)

The acute angle between the lines is 40.7° (1 d.p.)

1 c Direction vectors are $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}$

$\cos\theta = \dfrac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$\mathbf{a}\,.\,\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} = 2 - 7 + 3 = -2$

$|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$|\mathbf{b}| = \sqrt{2^2 + (-7)^2 + 3^2} = \sqrt{62}$

$\cos\theta = \dfrac{-2}{\sqrt{3}\sqrt{62}}$

$\theta = 98.4°$ (1 d.p.)

This is the angle between the two vectors.

The acute angle between the lines is 180° – 98.4° = 81.6° (1 d.p.).

d Direction vectors are $\mathbf{a} = \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}$

$\cos\theta = \dfrac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$\mathbf{a}\,.\,\mathbf{b} = \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} = 8 - 3 + 14 = 19$

$|\mathbf{a}| = \sqrt{8^2 + (-1)^2 + (-2)^2} = \sqrt{69}$

$|\mathbf{b}| = \sqrt{1^2 + 3^2 + (-7)^2} = \sqrt{59}$

$\cos\theta = \dfrac{19}{\sqrt{69}\sqrt{59}}$

$\theta = 72.7°$ (1 d.p.)

The acute angle between the lines is 72.7° (1 d.p.)

1 e Direction vectors are $\mathbf{a} = \begin{pmatrix} 11 \\ 5 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}\,.\,\mathbf{b} = \begin{pmatrix} 11 \\ 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} = -33 + 25 - 12 = -20$$

$$|\mathbf{a}| = \sqrt{11^2 + 5^2 + (-3)^2} = \sqrt{155}$$

$$|\mathbf{b}| = \sqrt{(-3)^2 + 5^2 + 4^2} = \sqrt{50}$$

$$\cos\theta = \frac{-20}{\sqrt{155}\sqrt{50}}$$

$\theta = 103.1°$ (1 d.p.)

This is the angle between the two vectors.

The acute angle between the lines is 180° – 103.1° = 76.9° (1 d.p.).

2 a $\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k}) = (\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})$

$= 2 - 1 - 1$

i.e. $\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k}) = 0$

b $\mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}-3\mathbf{k}) = (\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(5\mathbf{i}-\mathbf{j}-3\mathbf{k})$

$= 5 - 2 - 3$

i.e. $\mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}-3\mathbf{k}) = 0$

c $\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}+4\mathbf{k}) = (2\mathbf{i}-3\mathbf{k})\cdot(\mathbf{i}+3\mathbf{j}+4\mathbf{k})$

$= 2 - 12$

i.e. $\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}+4\mathbf{k}) = -10$

d $\mathbf{r}\cdot(4\mathbf{i}+\mathbf{j}-5\mathbf{k}) = (4\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(4\mathbf{i}+\mathbf{j}-5\mathbf{k})$

$= 16 - 2 - 5$

i.e. $\mathbf{r}\cdot(4\mathbf{i}+\mathbf{j}-5\mathbf{k}) = 9$

3 a $2x + y + z = 0$

Replace **r** by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in each equation.

b $5x - y - 3z = 0$

c $x + 3y + 4z = -10$

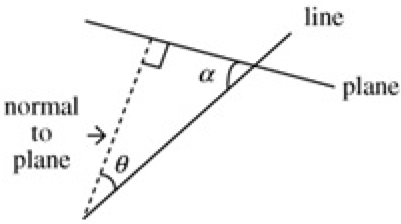
d $4x + y - 5z = 9$

4 Expanding the scalar product gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = k$$

$$\Rightarrow n_1 x + n_2 y + n_3 z = k$$

5



Find the acute angle between the given line and the normal to the plane, then subtract from 90°.

Let θ be the acute angle between the line and the normal to the plane.

$$\text{Then } \cos\theta = \left|\frac{(4\mathbf{i}+4\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{4^2+4^2+7^2}\sqrt{2^2+1^2+(-2)^2}}\right|$$

$$= \left|\frac{8+4-14}{\sqrt{81}\sqrt{9}}\right|$$

$$= \left|\frac{-2}{27}\right| = \frac{2}{27}$$

Let α be the angle between the line and the plane.

Then $\theta + \alpha = 90°$

So $\sin\alpha = \cos\theta = \dfrac{2}{27}$

$\therefore \alpha = 4.25°$ (3 s.f.)

6 Let θ be the acute angle between the line and the normal to the plane.

$$\text{Then } \cos\theta = \frac{(3\mathbf{i}+4\mathbf{j}-12\mathbf{k})\cdot(4\mathbf{i}-4\mathbf{j}-7\mathbf{k})}{\sqrt{3^2+4^2+(-12)^2}\sqrt{4^2+(-4)^2+(-7)^2}}$$

$$= \frac{12-16+84}{\sqrt{169}\sqrt{81}}$$

$$= \frac{80}{13\times 9}$$

$$= \frac{80}{117}$$

Find the acute angle between the given line and the normal to the plane, then subtract from 90°.

Let α be the angle between the line and the plane.

Then $\theta + \alpha = 90°$

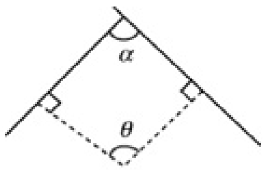
So $\sin\alpha = \cos\theta = \dfrac{80}{117}$

$\therefore \alpha = 43.1°$ (3s.f.)

7 Then angle θ between the two normal vectors $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}$ is given by

$$\cos\theta = \frac{(\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(-4\mathbf{i}+4\mathbf{j}+7\mathbf{k})}{|\mathbf{i}+2\mathbf{j}-2\mathbf{k}||-4\mathbf{i}+4\mathbf{j}+7\mathbf{k}|}$$
$$= \frac{-4+8-14}{\sqrt{1^2+2^2+(-2)^2}\sqrt{(-4)^2+4^2+7^2}}$$
$$= \frac{-10}{\sqrt{9}\sqrt{81}}$$
$$= -\frac{10}{27}$$

First find the angle between the two normal vectors.



The acute angle, α, between the two planes is such that $\alpha+\theta=180°$

So $\cos\alpha = -\cos\theta$
$$= \frac{10}{27}$$
$\therefore \alpha = 68.3°$ (3 s.f.)

8 The angle θ between the two normal vectors $3\mathbf{i}-4\mathbf{j}+12\mathbf{k}$ and $5\mathbf{i}-12\mathbf{k}$ is given by

$$\cos\theta = \frac{(3\mathbf{i}-4\mathbf{j}+12\mathbf{k})\cdot(5\mathbf{i}-12\mathbf{k})}{|3\mathbf{i}-4\mathbf{j}+12\mathbf{k}||5\mathbf{i}-12\mathbf{k}|}$$
$$= \frac{15-144}{\sqrt{3^2+(-4)^2+12^2}\sqrt{5^2+(-12)^2}}$$
$$= \frac{-129}{\sqrt{169}\sqrt{169}}$$
$$= \frac{-129}{169}$$

The acute angle α between the planes is such that $\alpha+\theta=180°$

So $\cos\alpha = -\cos\theta = \dfrac{129}{169}$

$\therefore \alpha = 40.2°$ (3 s.f.)

9 **a** Line l_1: $\mathbf{r} = \begin{pmatrix}1\\4\\2\end{pmatrix} + \lambda\begin{pmatrix}8\\5\\1\end{pmatrix}$

When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix}9\\9\\3\end{pmatrix}$

So the point (9, 9, 3) lies on l_1.

9 b Direction vectors are $\mathrm{a} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}$ and $\mathrm{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

$$\mathbf{a} \, . \, \mathbf{b} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = 24 + 5 + 0 = 29$$

$$|\mathrm{a}| = \sqrt{8^2 + 5^2 + 1^2} = \sqrt{90}$$

$$|\mathrm{b}| = \sqrt{3^2 + 1^2 + 0^2} = \sqrt{10}$$

$$\cos\theta = \frac{29}{\sqrt{90}\sqrt{10}} = \frac{29}{\sqrt{900}} = \frac{29}{30}$$

c $\mathbf{PQ} = \sqrt{(9-1)^2 + (9-4)^2 + (3-2)^2}$

$= \sqrt{8^2 + 5^2 + 1^2} = \sqrt{90}$

Line l_2: $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3s \\ 4+s \\ 2 \end{pmatrix}$

Let the coordinates of R be $(1+3s,\ 4+s,\ 2)$

$\mathbf{PR} = \sqrt{(1+3s-1)^2 + (4+s-4)^2 + (2-2)^2}$

$= \sqrt{9s^2 + s^2} = \sqrt{10s^2}$

$PQ^2 = PR^2 : 90 = 10s^2$

$\Rightarrow 5^2 = 9$

$\Rightarrow s = \pm 3$

When $s = 3$, $\mathbf{r} = \begin{pmatrix} 10 \\ 7 \\ 2 \end{pmatrix}$ $\quad R: (10,\ 7,\ 2)$

When $s = -3$, $\mathbf{r} = \begin{pmatrix} -8 \\ 1 \\ 2 \end{pmatrix}$ $\quad R: (-8,\ 1,\ 2)$

10 a $\frac{3-6}{-1} = \frac{3+3}{2} = \frac{7+2}{3} = 3$

and

$\frac{3+5}{2} = \frac{3-15}{-3} = \frac{7-3}{1} = 4$

So A lies on both l_1 and l_2

10 b $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$\therefore \mathbf{a} \cdot \mathbf{b} = -2 - 6 + 3 = -5$

$|\mathbf{a}| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$

$|\mathbf{b}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$

$\therefore \cos\theta = \left|\frac{-5}{\sqrt{14}\sqrt{14}}\right| = \frac{5}{14}$

$\Rightarrow \theta = 69.1^o$ (1d.p.)

11 $\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ 0 \end{pmatrix} \Rightarrow A = (10, 9, 0)$

$\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} - 2\begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \\ -4 \end{pmatrix} \Rightarrow B = (11, 11, -4)$

$\therefore \overrightarrow{AB} = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

l_1 has direction vector $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\mathbf{v}_1 \cdot \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 3 + 4 + 4 = 11$

$|\mathbf{v}_1| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21}$

$|\mathbf{v}_2| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$

$\therefore \cos\theta = \frac{11}{\sqrt{21}\sqrt{14}} \Rightarrow \theta = 50.1^o$ (1d.p.)

12 a Find the plane containing $A=(3,5,-1)$; $B=(2,-2,4)$; $C=(4,3,0)$ and show that $D=(1,4,-3)$ does not lie in it.

$\overrightarrow{AB}=\begin{pmatrix}-1\\-7\\5\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$

So an equation for the plane is

$$\mathbf{r}=\begin{pmatrix}3\\5\\-1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-7\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

If D lies in the plane, then there exist λ and μ such that

$$\begin{pmatrix}3\\5\\-1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-7\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}1\\4\\-3\end{pmatrix}$$

$\mathbf{i}$ component: $3-\lambda+\mu=1\Rightarrow-\lambda+\mu=-2$

$\mathbf{j}$ component: $5-7\lambda-2\mu=4\Rightarrow 7\lambda+2\mu=1$

$\mathbf{k}$ component: $-1+5\lambda+\mu=-3\Rightarrow 5\lambda+\mu=-2$

$\mathbf{i}$ and $\mathbf{j}$ components $\Rightarrow$ $\lambda=\frac{5}{9}$, $\mu=-\frac{13}{9}$

Check $\mathbf{k}$ component: $5\left(\frac{5}{9}\right)-\frac{13}{9}=\frac{4}{3}\neq-2$

So the equations are not consistent.

Therefore the points are not coplanar.

b $\overrightarrow{AD}=\mathbf{b}=\begin{pmatrix}-2\\-1\\-2\end{pmatrix}$

The plane has normal vector $\mathbf{n}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, since this is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\therefore \mathbf{b}\cdot\mathbf{n}=\begin{pmatrix}-2\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}=-10$$

$|\mathbf{b}|=\sqrt{(-2)^2+(-1)^2+(-2)^2}=3$

$|\mathbf{n}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$

$\Rightarrow \sin\theta=\left|\frac{-10}{3\sqrt{14}}\right|=\frac{10}{3\sqrt{14}}$

$\Rightarrow \theta=63.0^o$ (1d.p.)

13 Let E be the midpoint of AD. Then the coordinate of E is $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$

The angle θ between planes ABD and ADC is the same as the angle between the lines EB and EC

$$\overrightarrow{EB} = \overrightarrow{OB} - \overrightarrow{OE} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

$$\overrightarrow{EC} = \overrightarrow{OC} - \overrightarrow{OE} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$\overrightarrow{EB}.\overrightarrow{EC} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix} . \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} = -\frac{1}{4} + 1 - \frac{1}{4} = \frac{1}{2} \qquad (1)$$

and $\overrightarrow{EB}.\overrightarrow{EC} = \left|\begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}\right| \left|\begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}\right| \cos\theta = \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + \left(\frac{1}{2}\right)^2} \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + \left(-\frac{1}{2}\right)^2} \cos\theta$

$$= \sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}\cos\theta = \frac{3}{2}\cos\theta \qquad (2)$$

So equating equations (1) and (2) gives:

$$\frac{1}{2} = \frac{3}{2}\cos\theta$$

So therefore $\cos\theta = \frac{1}{3}$

Since the tetrahedron is regular, the angle between any two adjacent faces will be the same, i.e. $\arccos\left(\frac{1}{3}\right)$

14 Let point F be the point of the top of the pole.

So the coordinate of F is $(0,0,20)$

Let the points of the bases of the ropes be $A=(0,8,2)$, $B=(12,-5,3)$ and $C=(-2,6,5)$ respectively.

Then $\overrightarrow{FA}=\begin{pmatrix}0\\8\\-18\end{pmatrix}$, and $\overrightarrow{FC}=\begin{pmatrix}-2\\6\\-15\end{pmatrix}$

Let θ be the angle between FA and FC

$$\overrightarrow{FA}.\overrightarrow{FC}=\begin{pmatrix}0\\8\\-18\end{pmatrix}.\begin{pmatrix}-2\\6\\-15\end{pmatrix}=(0\times-2)+(8\times6)+(-18\times-15)=318$$

$$\left|\begin{pmatrix}0\\8\\-18\end{pmatrix}\right|=\sqrt{0^2+8^2+(-18)^2}=\sqrt{388} \text{ and } \left|\begin{pmatrix}-2\\6\\-15\end{pmatrix}\right|=\sqrt{(-2)^2+6^2+(-15)^2}=\sqrt{265}$$

So $\cos\theta=\dfrac{318}{\sqrt{388}\sqrt{265}}=0.9917$

and $\theta=\arccos(0.9917)=7.4^\circ$

So the angle between at least one pair of guide ropes is less than 15°. Therefore the flagpole will not be stable.