## Matrices 6E

**1 a** Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix}$$
$$= 1(4-1) - 0 + 0 = 3$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$
$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

**b** By inspection

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

**c** Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{-4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} - 0\begin{vmatrix} 0 & -\frac{4}{5} \\ 0 & \frac{3}{5} \end{vmatrix} + 0\begin{vmatrix} 0 & \frac{3}{5} \\ 0 & \frac{4}{5} \end{vmatrix}$$
$$= 1\left(\frac{9}{25} + \frac{6}{25}\right) - 0 + 0 = 1$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 0 & -\frac{4}{5} \\ 0 & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 0 & -\frac{3}{5} \\ 0 & \frac{4}{5} \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{4}{5} \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ \frac{3}{5} & -\frac{4}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -\frac{4}{5} \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

**1 c cont.** The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{-4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

**2 a** Let $\mathbf{A} = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$

$$\det(\mathbf{A}) = 1\begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix}$$
$$= 1(-4-0) + 3(0-3) + 2(0+6)$$
$$= -4 - 9 + 12 = -1$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} -3 & 2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 0 & -2 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} -4 & -3 & 6 \\ -6 & -4 & 9 \\ 1 & 1 & -2 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -4 & 3 & 6 \\ 6 & -4 & -9 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -4 & 6 & 1 \\ 3 & -4 & -1 \\ 6 & -9 & -2 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathrm{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} -4 & 6 & 1 \\ 3 & -4 & -1 \\ 6 & -9 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & -6 & -1 \\ -3 & 4 & 1 \\ -6 & 9 & 2 \end{pmatrix}$$

**2 b** Let $\mathbf{A} = \begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

$$\det(\mathbf{A}) = 2\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}$$
$$= 2(-2-1) - 3(3-2) + 2(3+4)$$
$$= -6 - 3 + 14 = 5$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} -3 & 1 & 7 \\ 1 & -2 & -4 \\ 7 & -4 & -13 \end{pmatrix}$$

**2 b cont.**

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -3 & -1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{5}\begin{pmatrix} -3 & 1 & 7 \\ -1 & -2 & 4 \\ 7 & 4 & -13 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{5} & -\frac{1}{5} & \frac{7}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{4}{5} & \frac{-13}{5} \end{pmatrix}$$

**c** Let

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\det(\mathbf{A}) = 3\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + (-7)\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}$$
$$= 3(6-2) - 2(-2-0) - 7(2-0)$$
$$= 12 + 4 - 14 = 2$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & -7 \\ -3 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -7 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 2 \\ 10 & -6 & 6 \\ -19 & 10 & -11 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 4 & 2 & 2 \\ -10 & -6 & -6 \\ -19 & -10 & -11 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix}$$

$$\mathbf{A}^{-1}\frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} 4 & -10 & -19 \\ 2 & -6 & -10 \\ 2 & -6 & -11 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -5 & -\frac{19}{2} \\ 1 & -3 & -5 \\ 1 & -3 & -\frac{11}{2} \end{pmatrix}$$

**3 a** $\det(\mathbf{A}) = 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}$

$$= 1 - 0 - 2 = -1$$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

**3 a cont.** The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

**b** $\det(\mathbf{B}) = 2\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$

$= -4 - 0 - 2 = -6$

The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 2 \\ 3 & 3 & 3 \\ 1 & 3 & -1 \end{pmatrix}$$

The matrix of cofactors is given by

$$\mathbf{C} = \begin{pmatrix} -2 & 0 & 2 \\ -3 & 3 & -3 \\ 1 & -3 & -1 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})}\mathbf{C}^{\mathrm{T}} = \frac{1}{-6}\begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

**c** $B^{1}A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -\frac{1}{3}+0-\frac{1}{3} & 0+\frac{1}{2}+0 & \frac{1}{3}+0+\frac{1}{6} \\ 0+0+1 & 0-\frac{1}{2}+0 & 0+0-\frac{1}{2} \\ \frac{1}{3}+0+\frac{1}{3} & 0+\frac{1}{2}+0 & -\frac{1}{3}+0-\frac{1}{6} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = (\mathbf{AB})^{-1}, \text{ as required.}$$

**4 a** $\det(\mathbf{A}) = 2\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} k & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix}$

$= 2(4-1) - 0 + 3(k-1)$

$= 6 + 3k - 3 = 3k + 3 = 3(k+1),$

as required.

**b** The matrix of minors is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ k & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ k & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4k-1 & k-1 \\ -3 & 5 & 2 \\ -3 & 2-3k & 2 \end{pmatrix}$$

**4 b cont.** The matrix of cofactors is given by

$$\mathbf{C}=\begin{pmatrix}3 & 1-4k & k-1\\ -3 & 5 & -2\\ -3 & 3k-2 & 2\end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}3 & 3 & -3\\ 1-4k & 5 & 3k-2\\ k-1 & -2 & 2\end{pmatrix}$$

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}=\frac{1}{3(k+1)}\begin{pmatrix}3 & 3 & -3\\ 1-4k & 5 & 3k-2\\ k-1 & -2 & 2\end{pmatrix}$$

**5** $\mathbf{A}=\mathbf{A}^{-1}$

Multiplying throughout by $\mathbf{A}$

$\mathbf{AA}=\mathbf{AA}^{-1}$

$\mathbf{A}^2=\mathbf{I}$

$$\mathbf{A}^2=\begin{pmatrix}5 & a & 4\\ b & -7 & 8\\ 2 & -2 & c\end{pmatrix}\begin{pmatrix}5 & a & 4\\ b & -7 & 8\\ 2 & -2 & c\end{pmatrix}$$

$$=\begin{pmatrix}ab+33 & -2a-8 & 8a+4c+20\\ 16-2b & ab+33 & 4b+8c-56\\ -2b+2c+10 & 2a-2c+14 & c^2-8\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Equating the second elements in the first row
$-2a-8=0\Rightarrow a=-4$

Equating the first elements in the second row

$16-2b=0\Rightarrow b=8$

Equating the first elements in the third row using $b=8$

$-2b+2c+10=0\Rightarrow -16+2c+10=0$

$2c=6\Rightarrow c=3$

$a=-4, b=8, c=3$

**6 a**

$$\mathbf{A}^2=\begin{pmatrix}2 & -1 & 1\\ 4 & -3 & 0\\ -3 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & -1 & 1\\ 4 & -3 & 0\\ -3 & 3 & 1\end{pmatrix}$$

$$=\begin{pmatrix}4-4-3 & -2+3+3 & 2+0+1\\ 8-12+0 & -4+9+0 & 4+0+0\\ -6+12-3 & 3-9+3 & -3+0+1\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 4 & 3\\ -4 & 5 & 4\\ 3 & -3 & -2\end{pmatrix}$$

$\mathbf{A}^3=\mathbf{A}^2\mathbf{A}$

$$=\begin{pmatrix}-3 & 4 & 3\\ -4 & 5 & 4\\ 3 & -3 & -2\end{pmatrix}\begin{pmatrix}2 & -1 & 1\\ 4 & -3 & 0\\ -3 & 3 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-6+16-9 & 3-12+9 & -3+0+3\\ -8+20-12 & 4-15+12 & -4+0+4\\ 6-12+6 & -3+9-6 & 3+0-2\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathbf{I}, \text{as required.}$$

**b** $\mathbf{A}^3=\mathbf{AA}^2=\mathbf{I}$
Comparing with the definition of an inverse
$\mathbf{AA}^{-1}=\mathbf{I}$

$$\mathbf{A}^{-1}=\mathbf{A}^2=\begin{pmatrix}-3 & 4 & 3\\ -4 & 5 & 4\\ 3 & -3 & -2\end{pmatrix}$$

**7 a** $\mathbf{A}^2=\begin{pmatrix}1&1&0\\3&-3&1\\0&3&2\end{pmatrix}\begin{pmatrix}1&1&0\\3&-3&1\\0&3&2\end{pmatrix}$

$=\begin{pmatrix}1+3+0&1-3+0&0+1+0\\3-9+0&3+9+3&0-3+2\\0+9+0&0-9+6&0+3+4\end{pmatrix}$

$=\begin{pmatrix}4&-2&1\\-6&15&-1\\9&-3&7\end{pmatrix}$

$\mathbf{A}^3=\mathbf{A}^2\mathbf{A}=\begin{pmatrix}4&-2&1\\-6&15&-1\\9&-3&7\end{pmatrix}\begin{pmatrix}1&1&0\\3&-3&1\\0&3&2\end{pmatrix}$

$=\begin{pmatrix}4-6+0&4+6+3&0-2+2\\-6+45+0&-6-45-3&0+15-2\\9-9+0&9+9+21&0-3+14\end{pmatrix}$

$=\begin{pmatrix}-2&13&0\\39&-54&13\\0&39&11\end{pmatrix}$

$13\mathbf{A}-15\mathbf{I}=\begin{pmatrix}13&13&0\\39&-39&13\\0&39&26\end{pmatrix}-\begin{pmatrix}15&0&0\\0&15&0\\0&0&15\end{pmatrix}$

$=\begin{pmatrix}-2&13&0\\39&-54&13\\0&39&11\end{pmatrix}=\mathbf{A}^3$

Hence

$\mathbf{A}^3=13\mathbf{A}+15\mathbf{I}$, as required.

**b** Multiply the result of part a throughout by $\mathbf{A}^{-1}$

$\mathbf{A}^3\mathbf{A}^{-1}=13\mathbf{A}\mathbf{A}^{-1}-15\mathbf{I}\mathbf{A}^{-1}$

$\mathbf{A}^2=13\mathbf{I}-15\mathbf{A}^{-1}$

Rearranging

$15\mathbf{A}^{-1}=13\mathbf{I}-\mathbf{A}^2$, as required

**c** Using the result of part **b**

$15\mathbf{A}^{-1}=13\mathbf{I}-\mathbf{A}^2=\begin{pmatrix}13&0&0\\0&13&0\\0&0&13\end{pmatrix}-\begin{pmatrix}4&-2&1\\-6&15&-1\\9&-3&7\end{pmatrix}$

$=\begin{pmatrix}9&2&-1\\6&-2&1\\-9&3&6\end{pmatrix}$

Hence

$\mathbf{A}^{-1}=\frac{1}{15}\begin{pmatrix}9&2&-1\\6&-2&1\\-9&3&6\end{pmatrix}$

**8 a** $\det(\mathbf{A})=2\begin{vmatrix}3&-2\\3&-4\end{vmatrix}-0\begin{vmatrix}4&-2\\0&-4\end{vmatrix}+1\begin{vmatrix}4&3\\0&3\end{vmatrix}$

$=2(-12+6)-0+1(12-0)$

$=-12+12=0$

Hence **A** is singular.

**b** The matrix of minors is given by

$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}3&-2\\3&-4\end{vmatrix}&\begin{vmatrix}4&-2\\0&-4\end{vmatrix}&\begin{vmatrix}4&3\\0&3\end{vmatrix}\\\begin{vmatrix}0&1\\3&-4\end{vmatrix}&\begin{vmatrix}2&1\\0&-4\end{vmatrix}&\begin{vmatrix}2&0\\0&3\end{vmatrix}\\\begin{vmatrix}0&1\\3&-2\end{vmatrix}&\begin{vmatrix}2&1\\4&-2\end{vmatrix}&\begin{vmatrix}2&0\\4&3\end{vmatrix}\end{pmatrix}$

$=\begin{pmatrix}-6&-16&12\\-3&-8&6\\-3&-8&6\end{pmatrix}$

The matrix of cofactors is given by

$\mathbf{C}=\begin{pmatrix}-6&16&12\\3&-8&-6\\-3&8&6\end{pmatrix}$

**8 c** $\mathbf{AC}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{pmatrix}\begin{pmatrix} -6 & 3 & -3 \\ 16 & -8 & 8 \\ 12 & -6 & 6 \end{pmatrix}$

$$= \begin{pmatrix} -12+0+12 & 6+0-6 & -6+0+6 \\ -24+48-24 & 12-24+12 & -12+24-12 \\ 0+48-48 & 0-24+24 & 0+24-24 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \text{ as required.}$$

**9 a** All values of $k$.

**b** The matrix of minors is given by

$$\begin{pmatrix} 2 & k & 3 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\det \mathbf{M} = 2\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} - k\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= -2 - 3 = -5$$

Matrix of minors

$$= \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} k & 3 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & k \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} k & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & k \\ -1 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & -1 \\ 3-k & -5 & -2-k \\ k-6 & 5 & 4+k \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -1 & 0 & -1 \\ k-3 & -5 & k+2 \\ k-6 & -5 & k+4 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -1 & k-3 & k-6 \\ 0 & -5 & -5 \\ -1 & k+2 & k+4 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{5}\begin{pmatrix} 1 & 3-k & 6-k \\ 0 & 5 & 5 \\ 1 & -k-2 & -k-4 \end{pmatrix}$$

**10** $\mathbf{A} = \begin{pmatrix} p & 2p & 3 \\ 4 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$

$$\det \mathbf{A} = p\begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} - 2p\begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 4 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= -2p + 2p + 21 = 4p - 21$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2p & 3 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} p & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} p & 2p \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2p & 3 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} p & 3 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} p & 2p \\ 4 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -7 \\ 6 & -3 & -4p \\ 2p+3 & p-12 & -p-8p \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 2 & 1 & -7 \\ -6 & -3 & 4p \\ 2p+3 & 12-p & -9p \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 2 & -6 & 2p+3 \\ 1 & -3 & 12-p \\ -7 & 4p & -9p \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{4p-21}\begin{pmatrix} 2 & -6 & 2p+3 \\ 1 & -3 & 12-p \\ -7 & 4p & -9p \end{pmatrix}$$