Review exercise 1

1 a $z_1 - z_2$

$= 4 - 5i - pi$

$= 4 - (5 + p)i$

b $z_1 z_2$

$= (4 - 5i)pi$

$= 4pi - 5pi^2$

$= 4pi + 5p$

$= 5p + 4pi$

c $\frac{z_1}{z_2}$

$= \frac{4 - 5i}{pi}$

$= \frac{i(4 - 5i)}{-p}$

$= \frac{4i + 5}{-p}$

$= -\frac{5}{p} - \frac{4}{p}i$

2 a $z^3 - kz^2 + 3z$

$= z(z^2 - kz + 3)$

So if there are 2 imaginary roots, the

discriminant of $z^2 - kz + 3 < 0$

$\Rightarrow (-k)^2 - 12 < 0$

$k^2 < 12$

$-2\sqrt{3} < k < 2\sqrt{3}$

b $z^3 - 2z^2 + 3z = 0$

$\Rightarrow z(z^2 - 2z + 3) = 0$

$\Rightarrow z = 0, z = \frac{2 \pm \sqrt{-8}}{2}$

$\Rightarrow z = 0, z = 1 \pm i\sqrt{2}$

3

$$z = \frac{5 \pm \sqrt{25-52}}{2}$$
$$= \frac{5 \pm \sqrt{-27}}{2}$$
$$= \frac{5}{2} \pm \frac{3\sqrt{3}}{2} i$$

So $z_1, z_2 = \frac{5}{2} + \frac{3\sqrt{3}}{2} i, \frac{5}{2} - \frac{3\sqrt{3}}{2} i$

4 $(2-i)x - (1+3i)y - 7 = 0$

$\Rightarrow (2x - y - 7) + (-x - 3y)\mathrm{i} = 0$

$\Rightarrow 2x - y = 7, x + 3y = 0$

$\Rightarrow x = 3, y = -1$

5 **a** $\frac{2+3\mathrm{i}}{5+\mathrm{i}} \times \frac{5-\mathrm{i}}{5-\mathrm{i}} = \frac{10 - 2\mathrm{i} + 15\mathrm{i} + 3}{26}$

$= \frac{13 + 13\mathrm{i}}{26} = \frac{1}{2} + \frac{1}{2}\mathrm{i}$

$= \frac{1}{2}(1+\mathrm{i})$

$\lambda = \frac{1}{2}$

$(5+\mathrm{i})(5-\mathrm{i}) = 5^2 + 1^2 = 26$ You should practise doing such calculations mentally.

You use the result from part **a** to simplify the working in part **b**.

b $\left(\frac{2+3\mathrm{i}}{5+\mathrm{i}}\right)^4 = \left[\frac{1}{2}(1+\mathrm{i})\right]^4$

$= \frac{1}{16}(1 + 4\mathrm{i} + 6\mathrm{i}^2 + 4\mathrm{i}^3 + \mathrm{i}^4)$

$= \frac{1}{16}(1 + 4\mathrm{i} - 6 - 4\mathrm{i} + 1)$

$= \frac{1}{16} \times -4 = -\frac{1}{4}$, a real number

$(1+\mathrm{i})^4$ is expanded using the binomial expansion

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$\mathrm{i}^3 = \mathrm{i}^2 \times \mathrm{i} = -1 \times \mathrm{i} = -\mathrm{i}$

$\mathrm{i}^4 = \mathrm{i}^2 \times \mathrm{i}^2 = -1 \times -1 = 1$

6 $-1+i$ is a root $\Rightarrow -1-i$ is also a root

$\Rightarrow (z+1-i)(z+1+i)$ is a factor

$\Rightarrow z^2 + 2z + 2$ is a factor

$\Rightarrow z^3 + 5z^2 + 8z + 6 = (z^2 + 2z + 2)(z+3)$

$\Rightarrow z = -3, -1 \pm i$

7 **a** $f(2-3i) = 0$

$\Rightarrow (2-3i)^3 - 6(2-3i)^2 + k(2-3i) - 26 = 0$

$\Rightarrow 8 - 36i - 54 + 27i - 24 + 72i + 54 + 2k - 3ki - 26 = 0$

Equating real coefficients $-42 + 2k = 0 \Rightarrow k = 21$

7 b $2+3i$ must also be a factor

$\Rightarrow (z-2+3i)(z-2-3i) = z^2-4z+13$ is a factor

$\Rightarrow z^3-6z^2+21z-26 = (z^2-4z+13)(z-2)$

$\Rightarrow z = 2, 2+3i$ are the other two factors

8 a $b-3=-1 \Rightarrow b=2$

$-4c=-16 \Rightarrow c=4$

$\Rightarrow z^4-z^3-6z^2-20z-16 = (z^2-3z-4)(z^2+2z+4)$

b $z^4-z^3-6z^2-20z-16 = (z-4)(z+1)(z^2+2z+4)$

$\Rightarrow z = 4, -1, \dfrac{-2\pm\sqrt{12}}{2}$

$\Rightarrow z = 4, -1, -1\pm\sqrt{3}i$

9 $(z-1-2i)(z-1+2i)$ must be a factor

$\Rightarrow z^2-2z+5$ is a factor

$\Rightarrow z^4-8z^3+27z^2-50z+50$

$=(z^2-2z+5)(z^2+kz+10)$

Equating coefficients of z^3

$-2+k=-8 \Rightarrow k=-6$

$\Rightarrow (z^2-2z+5)(z^2-6z+10)=0$

$\Rightarrow z = 1\pm 2i, \dfrac{6\pm\sqrt{-4}}{2}$

$\Rightarrow z = 1\pm 2i, 3\pm i$

10 a Comparing constant coefficients

$\alpha \times \dfrac{4}{\alpha} \times (\alpha + \dfrac{4}{\alpha} + 1) = 12$

$\Rightarrow 4(\alpha + \dfrac{4}{\alpha} + 1) = 12$

$\Rightarrow \alpha^2 + 4 + \alpha = 3\alpha$

$\Rightarrow \alpha^2 - 2\alpha + 4 = 0$

$\Rightarrow \alpha = 1\pm\sqrt{3}i$

So the roots are $1\pm\sqrt{3}i, 3$

b $f(z) = (z-3)(z-1-\sqrt{3}i)(z-1-\sqrt{3}i)$

$=(z-3)(z^2-2z+4)$

$=z^3-5z^2+10z-12$

$\Rightarrow p=-5, q=10$

11 a $\dfrac{3z-1}{2-\mathrm{i}}=\dfrac{4}{1+2\mathrm{i}}$

$$3z-1=\frac{8-4\mathrm{i}}{1+2\mathrm{i}}\times\frac{1-2\mathrm{i}}{1-2\mathrm{i}}$$

$$=\frac{8-16\mathrm{i}-4\mathrm{i}-8}{5}=\frac{-20\mathrm{i}}{5}=-4\mathrm{i}$$

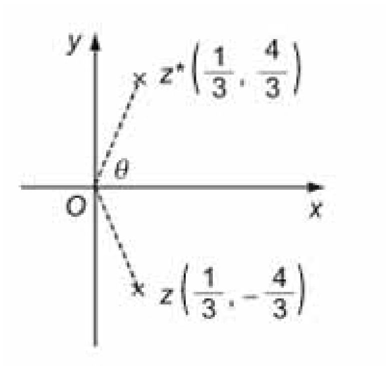
$$3z=1-4\mathrm{i}$$

$$z=\frac{1}{3}-\frac{4}{3}\mathrm{i}$$

You multiply both sides of the equation by $2-\mathrm{i}$.

Then multiply the numerator and denominator by the conjugate complex of the denominator.

b



You place the points in the Argand diagram which represent conjugate complex numbers symmetrically about the real x-axis.

Label the points so it is clear which is the original number $(\mathbf{z})$ and which is the conjugate $(\mathbf{z}^*)$.

c $|z|^2=\left(\dfrac{1}{3}\right)^2+\left(-\dfrac{4}{3}\right)^3=\dfrac{1}{9}+\dfrac{16}{9}=\dfrac{17}{9}$

$$|z|=\frac{\sqrt{17}}{3}$$

$$\tan\theta=\frac{\frac{4}{3}}{\frac{1}{3}}=4\Rightarrow\theta\approx76^\circ$$

z is in the fourth quadrant.

$\arg z=-76^\circ$, to the nearest degree.

$$z=\frac{\sqrt{17}}{3}\cos(-76^\circ)+\mathrm{i}\frac{\sqrt{17}}{3}\sin(-76^\circ)$$

$$z^*=\frac{\sqrt{17}}{3}\cos76^\circ+\mathrm{i}\frac{\sqrt{17}}{3}\sin76^\circ$$

The diagram you have drawn in part **b** shows that z is in the fourth quadrant. There is no need to draw it again.

It is always true $|z^*|=|z|$ and $\arg z^*=-\arg z$, so you just write down the final answer without further working.

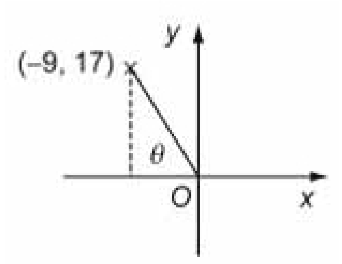
12 z lies on a circle radius 1, centre $4i$

So the maximum and minimum arguments lie on the tangents to the circle from O

$\Rightarrow$ minimum = $\arctan(\sqrt{15})=1.318$

maximum = $\pi-\arctan(\sqrt{15})=1.823$

13 a



b $\tan\theta = \frac{17}{9} \Rightarrow \theta = 1.084...$

You have to give your answer to 2 decimal places. To do this accurately you must work to at least 3 decimal places. This avoids rounding errors and errors due to premature approximation.

z is in the second quadrant.

$\arg z = \pi - 1.084\ldots = 2.057\ldots$

$= 2.06$, in radians to 2 d.p.

c $w = \frac{25+35\mathrm{i}}{z} = \frac{25+35\mathrm{i}}{-9+17\mathrm{i}} = \frac{25+35\mathrm{i}}{-9+17\mathrm{i}} \times \frac{-9-17\mathrm{i}}{-9-17\mathrm{i}}$

$= \frac{-225-425\mathrm{i}-315\mathrm{i}+595}{(-9)^2+17^2}$

$= \frac{370-740\mathrm{i}}{370} = 1-2\mathrm{i}$

In this question, the arithmetic gets complicated. Use a calculator to help you with this. However, when you use a calculator, remember to show sufficient working to make your method clear.

14 a $|z_1|^2 = 5^2 + 1^2 = 26$

$|z_2|^2 = (-2)^2 + 3^2 = 4 + 9 = 13$

$26 = 2 \times 13$

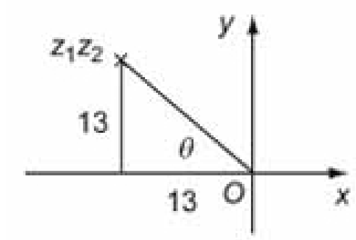
Hence $|z_1|^2 = 2|z_2|^2$, as required.

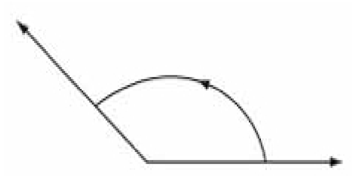
If $z = a + \mathrm{i}b$, then $|z|^2 = a^2 + b^2$

When you are asked to show or prove a result, you should conclude by saying that you have proved or shown the result. You can write the traditional q.e.d. if you like!

b $z_1 z_2 = (5+\mathrm{i})(-2+3\mathrm{i})$

$= -10 + 15\mathrm{i} - 2\mathrm{i} - 3 = -13 + 13\mathrm{i}$





The argument is the angle with the positive x-axis. Anti-clockwise is positive.

$\tan\theta = \frac{13}{13} = 1 \Rightarrow \theta = \frac{\pi}{4}$

$z_1 z_2$ is in the second quadrant.

$\arg(z_1 z_2) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

As the question has not specified that you should work in radians or degrees, you could work in either and 135° would also be an acceptable answer.

15 a $z^2 = (2+i)^2 = 4 - 4i + i^2$
$= 4 - 4i - 1$
$= 3 - 4i$, as required.

You square using the formula $(a-b)^2 = a^2 - 2ab + b^2$

b From part **a**, the square roots of $3-4i$ are $2-i$ and $-2+i$.

Taking square roots of both sides of the equation $(z+i)^2 = 3-4i$

$z + i = 2 - i \Rightarrow z = 2 - 2i$
$z + i = -2 + i \Rightarrow z = -2$

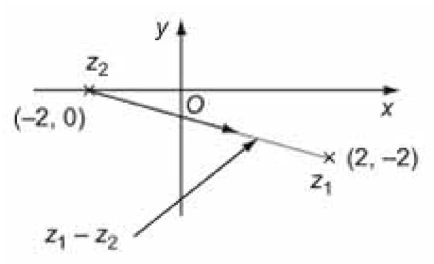
$z_1 = 2 - 2i$, say, and $z_2 = -2$

The square root of any number *k*, real or complex, is a root of $z^2 = k$. Hence, part **a** shows that one square root of $3-4i$ is $2-i$.

If one square root of $3-4i$ is $2-i$, then the other is $-(2-i)$.

z_1 and z_2 could be the other way round but that would make no difference to $|z_1 - z_2|$ or $z_1 - z_2$, the expressions you are asked about in parts **d** and **e**.

c



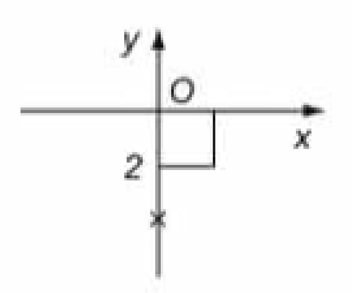
$z_1 - z_2$ can be represented on the diagram you drew in part **c** by the vector joining the point representing z_1 to the point representing z_2. The modulus of $z_1 - z_2$ is then just the length of the line joining these two points and this length can be found using coordinate geometry.

d Using the formula

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
$= (2-(-2))^2 + (-2-0)^2$
$= 4^2 + 2^2 = 20$

Hence $|z_1 - z_2| = \sqrt{20} = 2\sqrt{5}$

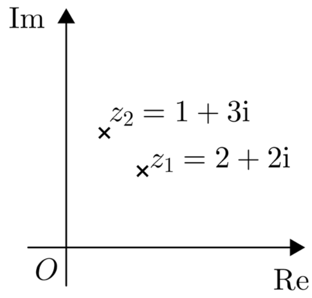
e $z_1 + z_2 = 2 - 2i - 2 = -2i$



$\arg(z_1 + z_2) = -\frac{\pi}{2}$

The argument of any number on the negative imaginary axis is $-\frac{\pi}{2}$ or $-90°$.

16 a



b $|z_1|^2 = 2^2 + 2^2 = 8 = 4 \times 2 \Rightarrow |z_1| = 2\sqrt{2}$

$|z_2|^2 = 1^2 + 3^2 = 10 \Rightarrow |z_2| = \sqrt{10}$

P has coordinates (2, 2) and Q (1, 3)

$PQ^2 = (1-2)^2 + (3-2)^2 = (-1)^2 + 1^2 = 2$

$PQ = \sqrt{2}$

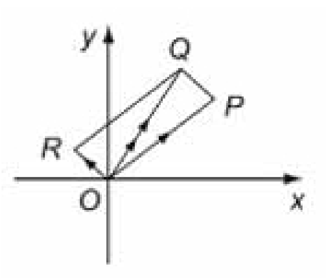
You use the formula $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ from Coordinate Geometry to calculate PQ^2.

c From **b**, $OP = 2\sqrt{2}$ and $OQ = \sqrt{10}$.

$$OP^2 + PQ^2 = (2\sqrt{2})^2 + (\sqrt{2})^2$$
$$= 8 + 2 = 10$$
$$= OQ^2$$

By the converse of Pythagoras' Theorem, ΔOPQ is right-angled.

d



You use the representation of the addition of complex numbers in an Argand diagram. The diagonal OQ of the parallelogram represents the addition of the two adjacent sides, OP and OR, of the parallelogram. (A rectangle is a special case of a parallelogram).

$\overline{OP} + \overline{OR} = \overline{OQ}$

$2 + 2i + z_3 = 1 + 3i$

$z_3 = -1 + i$

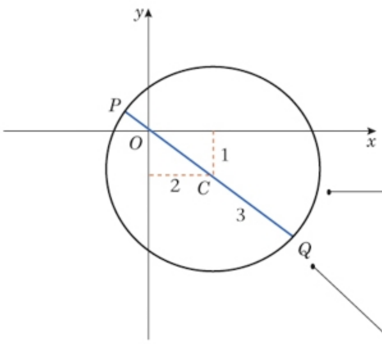
17

$$\frac{\cos 2x + i\sin 2x}{\cos 9x - i\sin 9x}$$
$$= \frac{(\cos 2x + i\sin 2x)(\cos 9x + i\sin 9x)}{(\cos 9x - i\sin 9x)(\cos 9x - i\sin 9x)}$$
$$= \frac{\cos 2x\cos 9x - \sin 2x\sin 9x + i(\sin 2x\cos 9x + \cos 2x\sin 9x)}{\cos^2 9x + \sin^2 9x}$$
$$= \cos 2x\cos 9x - \sin 2x\sin 9x + i(\sin 2x\cos 9x + \cos 2x\sin 9x)$$
$$= \cos 11x + i\sin 11x$$
$$n = 11$$

Use $\cos^2 A + \sin^2 A \equiv 1$

Use $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ and $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ with A=2x and B=9x

18 a



The locus of $|z-a| = k$, where a is a complex number and k is a real number, is a circle with radius k and centre the point representing a. Rewriting the relation in the question as $|z-(2-\mathrm{i})| = 3$, this locus is a circle of radius 3 with centre $(2, -1)$.

$|z|$ is the distance of the point representing z from the origin. The point on the circle furthest from O is marked by Q on the diagram and the point closest to O by P. The distances of Q and P from O represent the maximum and minimum values of $|z|$ respectively.

b $OC^2 = 1^2 + 2^2 = 5 \Rightarrow OC = \sqrt{5}$

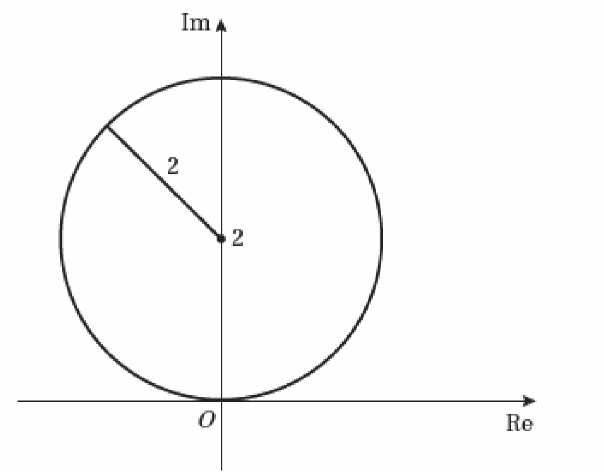
$OQ = OC + CQ = \sqrt{5} + 3$

Hence the maximum value of $|z|$ is $3 + \sqrt{5}$.

$OP = CP - CO = 3 - \sqrt{5}$

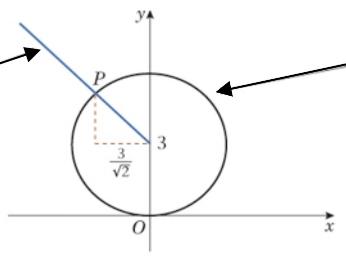
Hence the minimum value of $|z|$ is $3 - \sqrt{5}$.

19 a The locus of z is a circle of radius 2 and centre at $2i$



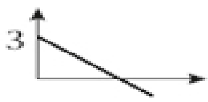
b $|z| \leqslant 2+2$ because it is the 3rd side of a triangle whose other sides are both 2. The maximum value of 4 occurs when $z = 4i$.

20 a



The locus of P is the circle with centre (0, 3) and radius 3. The coordinates (0, 3) represent the complex number 3i in the Argand diagram.

The half-line representing $\arg(z-3\text{i}) = \frac{3\pi}{4}$ has been added to the diagram. This starts at (0, 3) and makes an angle of $\frac{3\pi}{4}$ with the positive x-direction. It is a common error to turn this half into a Full line. The half line has a different equation,



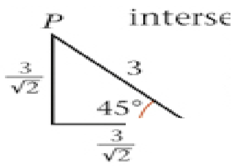
$\arg(z-3\text{i}) = -\frac{\pi}{4}$.

b From the diagram, z is the intersection of the circle and the half line marked P in the diagram.

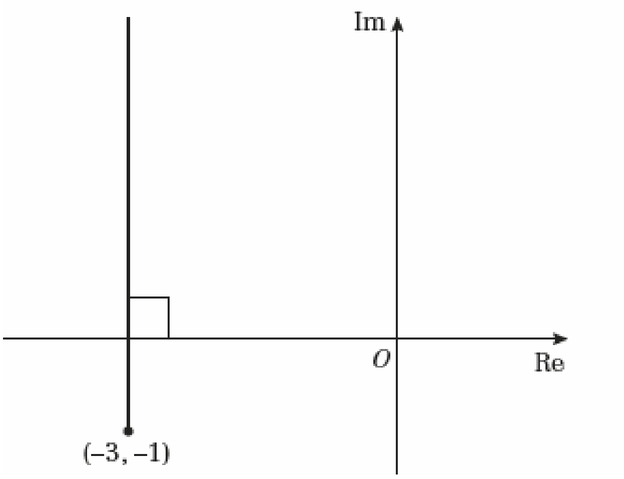
$$z = -\frac{3}{\sqrt{2}} + \text{i}\left(3 + \frac{3}{\sqrt{2}}\right)$$

$$= -\frac{3\sqrt{2}}{2} + \text{i}\left(3 + \frac{3\sqrt{2}}{2}\right)$$

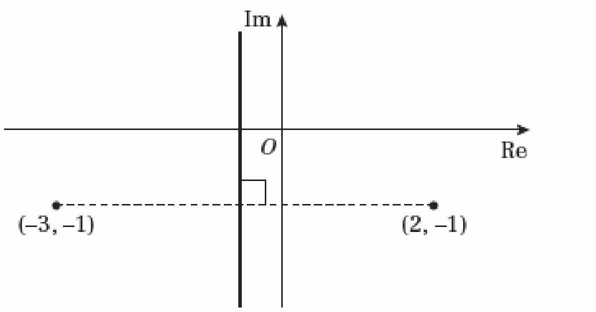
The geometry of the point of intersection is shown here. The coordinates of P can then be just written down.



21 $(z+3+i)=(z-(-3-i)$ so the locus is a vertical half line from the point $-3-i$.



22 a The locus is the perpendicular bisector of the line joining the points $-3-i$ and $2-i$



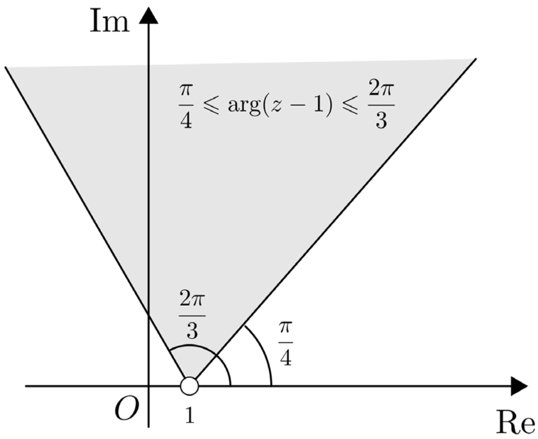
b The minimum value of $|z|$ is the minimum distance of O from the perpendicular bisector, so where the locus crosses the real axis. So minimum $=\frac{1}{2}$

c From the diagram, when $\arg z=-\frac{3\pi}{4}$

the point on the locus is $-\frac{1}{2}-\frac{1}{2}i$ because

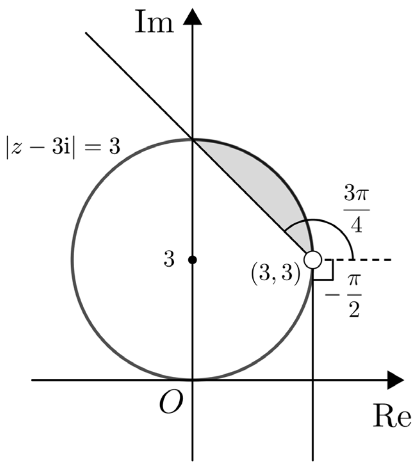
$\frac{x}{y}=\tan\left(-\frac{3\pi}{4}\right)=1$ and x must $=-\frac{1}{2}$

23 $\frac{\pi}{4} \leqslant \arg(z-1) \leqslant \frac{2\pi}{3}$ is the region between the half lines from z=1 making angles of $\frac{\pi}{4}$ and $\frac{2\pi}{3}$ with the real axis.



24 $-\frac{\pi}{2} < \arg(z-3-3i) \leqslant \frac{3\pi}{4}$ is the region between the half lines from 3+3i making angles of $-\frac{\pi}{2}, \frac{3\pi}{4}$ with the real axis.

$|z-3i| \leqslant 3$ is the inside of a circle centre 3i and radius 3.



25 $\sum_{r=1}^{n}(2r-1)^2=\sum_{r=1}^{n}(4r^2-4r+1)$

$=4\sum_{r=1}^{n}r^2-4\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$

$=\frac{4n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n$

$=\frac{2n(n+1)(2n+1)}{3}-\frac{6n(n+1)}{3}+\frac{3n}{3}$

$=\frac{n}{3}[2(n+1)(2n+1)-6(n+1)+3]$

$=\frac{n}{3}[4n^2+6n+2-6n-6+3]$

$=\frac{1}{3}n(4n^2-1)$, as required.

$\sum_{r=1}^{n}1=\underbrace{1+1+1+...+1}_{n\text{ times}}=n$

It is a common error to write $\sum_{r=1}^{n}1=1$.

After 'cancelling' the fractions, you should put all terms over a common denominator, here 3.

26 $\sum_{r=1}^{n}r(r^2-3)=\sum_{r=1}^{n}r^3-3\sum_{r=1}^{n}r$

$=\frac{n^2(n+1)^2}{4}-\frac{3n(n+1)}{2}$

$=\frac{n^2(n+1)^2}{4}-\frac{6n(n+1)}{4}$

$=\frac{n(n+1)}{4}[n(n+1)-6]$

$=\frac{n(n+1)}{4}[n^2+n-6]$

$=\frac{1}{4}n(n+1)(n-2)(n+3)$, as required.

After putting both terms over a common denominator, look for the common factors of the terms, here shown in **bold**;

$\frac{\mathbf{n}^2(\mathbf{n+1})^2}{\mathbf{4}}-\frac{6\mathbf{n}(\mathbf{n+1})}{\mathbf{4}}.$

You take these, together with the common denominator 4, outside a bracket;

$\frac{\mathbf{n(n+1)}}{\mathbf{4}}[n(n+1)-6].$

You need to be careful with the squared terms.

27 a $\sum_{r=1}^{n}r(2r-1)=\sum_{r=1}^{n}(2r^2-r)$

$=2\sum_{r=1}^{\pi}r^2-\sum_{r=1}^{\pi}r$

$=\frac{2n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}$

$=\frac{2n(n+1)(2n+1)}{6}-\frac{3n(n+1)}{6}$

$=\frac{n(n+1)}{6}[2(2n+1)-3]$

$=\frac{n(n+1)}{6}[4n+2-3]$

$=\frac{n(n+1)(4n-1)}{6}$, as required

You put the expressions over a common denominator, here 6, and then look for the common factors of the expressions, here n and $(n+1)$.

27 b $\sum_{r=1}^{30} r(2r-1) = \sum_{r=1}^{30} r(2r-1) - \sum_{r=1}^{10} r(2r-1)$

Substituting $n = 30$ and $n = 10$ into the result in part (a).

$$\sum_{r=11}^{30} r(2r-1) = \frac{30\times31\times119}{6} - \frac{10\times11\times39}{6}$$
$$= 18\,445 - 715$$
$$= 17\,730$$

> $\sum_{r=11}^{30} \mathrm{f}(r) = \sum_{r=1}^{30} \mathrm{f}(r) - \sum_{r=1}^{10} \mathrm{f}(r).$
>
> You find the sum from the 11th to the 30th term by subtracting the sum from the first to the 10th term from the sum from the first to the 30th term. It is a common error to subtract one term too many, in this case the 11th term. The sum you are finding starts with the 11th term. You must not subtract it from the series – you have to leave it in the series.

28 a $\sum_{r=1}^{n} (6r^2 + 4r - 5) = 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 5$

$$= \frac{\not{6}n(n+1)(2n+1)}{\not{6}} + \frac{\not{4}n(n+1)}{\not{2}} - 5n$$
$$= n(n+1)(2n+1) + 2n(n+1) - 5n$$
$$= n[(n+1)(2n+1) + 2(n+1) - 5]$$
$$= n[2n^2 + 3n + 1 + 2n + 2 - 5]$$
$$= n(2n^2 + 5n - 2), \text{ as required}$$

> A common error with the last term is to write $-\sum_{r=1}^{n} 5 = -5.$
>
> Correctly:
>
> $-\sum_{r=1}^{n} 5 = -(5+5+5+\ldots+5)$
>
> $= -5(\underbrace{1+1+1\ldots+1}_{n\text{ times}})$
>
> $= -5n$

b $\sum_{r=10}^{25} (6r^2 + 4r - 5) = \sum_{r=1}^{25} (6r^2 + 4r - 5) - \sum_{r=1}^{9} (6r^2 + 4r - 5)$

Substituting $n = 25$ and $n = 9$ into the result in part (a)

$\sum_{r=10}^{25} (6r^2 + 4r - 5)$

$= 25(2\times25^2 + 5\times25 - 2) - 9(2\times9^2 + 5\times9 - 2)$

$= 34\ 325 - 1845 = 32\ 480$

29 a $\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{3(n+1)}{6}$$
$$= \frac{n(n+1)}{6}[2n+1+3]$$
$$= \frac{n(n+1)\not{2}^{1}(n+2)}{\not{6}^{3}}$$
$$= \frac{1}{3}n(n+1)(n+2), \text{ as required.}$$

> After putting the expressions over a common denominator 6, you look for any factors common to both expressions. Here there are two, n and $(n+1)$.

29 b $\sum_{r=1}^{3n} r(r+1) = \sum_{r=1}^{3n} r(r+1) - \sum_{r=1}^{n-1} r(r+1)$

$= \frac{1}{3}3n(3n+1)(3n+2) - \frac{1}{3}(n-1)n(n+1)$

$= \frac{1}{3}n[3(3n+1)(3n+2) - (n-1)(n+1)]$

$= \frac{1}{3}n[27n^2 + 27n + 6 - (n^2 - 1)]$

$= \frac{1}{3}n(26n^2 + 27n + 7)$

$= \frac{1}{3}n(2n+1)(13n+7)$

$p = 13, q = 7$

To find an expression for $\sum_{r=1}^{n=1} r(r+1)$, you replace the n in the result in part (a) by $n - 1$;

$\frac{1}{3}n(n+1)(n+2)$

becomes $\frac{1}{3}(n-1)\big((n-1)+1\big)\big((n-1)+2\big)$

$= \frac{1}{3}(n-1)n(n+1)$.

As you are given that $(2n+1)$ is one factor of $26n^2 + 27n + 7$, the other can just be written down. $(2n+1)(pn+q) = 26n^2 + 27n + 7$, only if $2p = 26$ and $1q = 7$

30 a $\sum_{r=1}^{n} r^2(r-1) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2$

$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}$

$= \frac{3n^2(n+1)^2}{12} - \frac{2n(n+1)(2n+1)}{12}$

$= \frac{n(n+1)}{12}[3n(n+1) - 2(2n+1)]$

$= \frac{n(n+1)}{12}[3n^2 + 3n - 4n - 2]$

$= \frac{1}{12}n(n+1)(3n^2 - n - 2)$

$p = 3, q = -1, r = -2$

After putting the expressions over a common denominator 12, you look for any factors common to both expressions. Here there are two, n and $(n+1)$.

b $\sum_{r=50}^{100} r^2(r-1) = \sum_{r=1}^{100} r^2(r-1) - \sum_{r=1}^{49} r^2(r-1)$

$= \frac{1}{12}\times 100\times 101\times(3\times 100^2 - 100 - 2)$

$- \frac{1}{2}\times 49\times 50\times(3\times 49^2 - 49 - 2)$

$= 25\ 164150 - 1\ 460\ 200$

$= 23\ 703\ 950$

$\sum_{r=50}^{100} \mathrm{f}(r) = \sum_{r=1}^{100} \mathrm{f}(r) - \sum_{r=1}^{49} \mathrm{f}(r)$.

You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th Term from the sum from the first to the 100th term. It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not remove it from the series.

31 a $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{k}{3}$

$\Rightarrow k = -12$

b In the equation $ax^3 + bx^2 + cx + d = 0$,

$\alpha + \beta + \gamma = -\frac{b}{a} = 0$

$\alpha\beta\gamma = -\frac{d}{a} = -\frac{11}{3}$

c $(1-\alpha)(1-\beta)(1-\gamma)$

$= 1 - (\alpha + \beta + \gamma) + (\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma$

$= 1 - 0 - 4 + \frac{11}{3}$

$= \frac{2}{3}$

32 a $\alpha\beta\gamma\delta = \frac{d}{a}$ so $\frac{-4}{a} = -1 \Rightarrow a = 4$

b $\sum\alpha = -\frac{b}{a} = -\frac{7}{4}$

$\sum\alpha\beta = \frac{c}{a} = \frac{5}{4}$

$\sum\alpha\beta\gamma = -\frac{d}{a} = -\frac{3}{4}$

c $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2\sum\alpha\beta$

$= \left(-\frac{7}{4}\right)^2 - 2 \times \frac{5}{4}$

$= \frac{9}{16}$

33 Let $w = 2x + 1$

$\Rightarrow x = \frac{w-1}{2}$

$\left(\frac{w-1}{2}\right)^3 + 3\left(\frac{w-1}{2}\right)^2 + 5 \times \frac{w-1}{2} - 1 = 0$

$\Rightarrow \frac{w^3 - 3w^2 + 3w - 1}{8}$

$+ \frac{3(w^2 - 2w + 1)}{4} + 5 \times \frac{w+1}{2} - 1 = 0$

$\Rightarrow w^3 - 3w^2 + 3w - 1 + 6w^2 - 12w +$

$6 + 20w - 20 - 8 = 0$

$\Rightarrow w^3 + 3w^2 + 11w - 23 = 0$

34 a Let $w = 3x$

$\Rightarrow x = \frac{w}{3}$

$\left(\frac{w}{3}\right)^4 - \left(\frac{w}{3}\right)^3 - 2\left(\frac{w}{3}\right)^2 + 3\times\frac{w}{3} + 4 = 0$

$\Rightarrow \frac{w^4}{81} - \frac{w^3}{27} - \frac{2w^2}{9} + w + 4 = 0$

$\Rightarrow w^4 - 3w^3 - 18w^2 + 81w + 324 = 0$

b Let $w = 2x - 1$

$\Rightarrow x = \frac{w+1}{2}$

$\left(\frac{w+1}{2}\right)^4 - \left(\frac{w+1}{2}\right)^3 - 2\left(\frac{w+1}{2}\right)^2 + 3\times\frac{w+1}{2} + 4 = 0$

$\Rightarrow \frac{w^4 + 4w^3 + 6w^2 + 4w + 1}{16} - \frac{w^3 + 3w^2 + 3w + 1}{8}$

$- \frac{2(w^2 + 2w + 1)}{4} + 3\times\frac{w+1}{2} + 4 = 0$

$\Rightarrow w^4 + 4w^3 + 6w^2 + 4w + 1 - 2w^3 - 6w^2 - 6w -$

$2 - 8w^2 - 16w - 8 + 24w + 24 + 64 = 0$

$\Rightarrow w^4 + 2w^3 - 8w^2 + 6w + 79 = 0$

35 a Crosses x-axis at $x = a$ so

$a > 0$ and $a\sqrt{1 - a^2} = 0$

So $a = 1$

b $\pi\int_0^1 x^2(1 - x^2)dx$

$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$

$= \frac{2\pi}{15}$

36 $y = \sqrt{x^2+3} \Rightarrow y^2 = x^2+3$

$x^2 = y^2 - 3$

$$\text{Volume} = \pi\int_2^k (y^2-3)\,\mathrm{d}y$$

$$= \pi\left[\frac{1}{3}x^3 - 3x\right]_2^k$$

$$= \pi\left(\frac{1}{3}k^3 - 3k - \frac{8}{3} + 6\right)$$

$$= \pi\left(\frac{1}{3}k^3 - 3k + \frac{10}{3}\right)$$

So $\frac{1}{3}k^3 - 3k + \frac{10}{3} = 30$

$\Rightarrow k^3 - 9k + 10 = 90$

$\Rightarrow k^3 - 9k - 80 = 0$

$\Rightarrow (k-5)(k^2+5k+16) = 0$

$\Rightarrow k = 5$

37 Curve and line cross when $4 - x^2 = 2x + 1$

$\Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow (x+3)(x-1) = 0$

So $x = 1$

$$\text{So volume} = \pi\int_0^1 (2x+1)^2\mathrm{d}x + \pi\int_1^2 (4-x^2)^2\mathrm{d}x$$

$$= \pi\int_0^1 (4x^2+4x+1)^2\mathrm{d}x + \pi\int_1^2 (16-8x^2+x^4)^2\mathrm{d}x$$

$$= \pi\left[\frac{4}{3}x^3 + 2x^2 + x\right]_0^1 + \pi\left[16x - \frac{8}{3}x^3 + \frac{x^5}{5}\right]_1^2$$

$$= \pi\left(\frac{4}{3} + 2 + 1\right) + \pi\left(32 - \frac{64}{3} + \frac{32}{5} - 16 + \frac{8}{3} - \frac{1}{5}\right)$$

$$= 24.71\ldots \simeq 24.7$$

38 a $x^2 + (y-k)^2 = 100$ is the equation of a circle centre $(0, k)$ and radius 10 so $k = 20$

b $V = \pi\int_a^b 100 - (y-20)^2\,\mathrm{d}y = \pi\int_a^b 40y - y^2 - 300\,\mathrm{d}y$

$$= \pi\left[20y^2 - \frac{y^3}{3} - 300y\right]_a^b$$

$$= \frac{\pi}{3}\left(60\left(b^2 - a^2\right) - \left(b^3 - a^3\right) - 900\left(b - a\right)\right)$$

38 c Volume of stand

=Volume of cylinder - volume of cut out section

$=\pi\times10^2\times20-\frac{\pi}{3}\left(60(20^2-10^2)-(20^3-10^3)-900\times10\right)$

$=4188.79$

So cost $=4188.79\times.025$

$=$ £104.72 to nearest penny

39 a $y=12-x^2$ crosses the y-axis when y=12.

So the maximum outer radius =12 mm and maximum outer diameter =24 mm.

b Curves cross when $12-x^2=8-0.2x^2$

$\Rightarrow 0.8x^2=5$

$\Rightarrow x=\pm\sqrt{5}$

So volume $=\pi\int_{-\sqrt{5}}^{\sqrt{5}}(12-x^2)^2-(8-0.2x^2)\mathrm{d}x$

$=\pi\int_{-\sqrt{5}}^{\sqrt{5}}(80-20.8x^2+0.96x^4)\mathrm{d}x$

$=\pi\left[80x-\frac{20.8x^3}{3}+\frac{0.96x^5}{5}\right]_{-\sqrt{5}}^{\sqrt{5}}$

$=704.355$

So mass $=704.355\times19.3\div1000$

$=13.6\,\mathrm{g}$

c Any valid reason, e.g. the gold may have voids or impurities, the actual dimensions may differ from those modelled, answer is given to too great a degree of accuracy.

Challenge

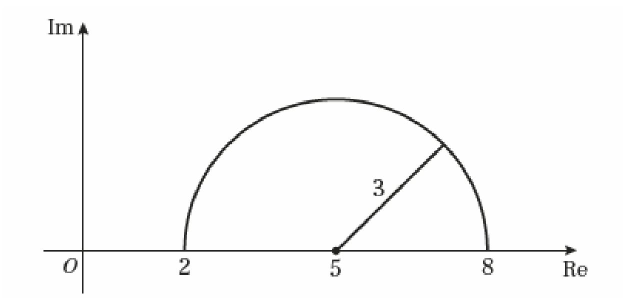
1 a $\arg\left(\frac{z-8}{z-2}\right)=\frac{\pi}{2}$

$\Rightarrow \arg(z-8)-\arg(z-2)=\frac{\pi}{2}$

So the angle between the lines joining z to 8

and 2 is $\frac{\pi}{2}$. So it is a semi-circle

radius 3 and centre 5.



Challenge

1 b $|z-5| =$ radius of semi-circle = 3.

2 a Let $u_r = ar + b$

$$\sum_{r=1}^{n} u_r = \frac{a}{2}n(n+1) + bn = n^2 + 5n$$

$$\Rightarrow a = 2 \text{ and } \frac{a}{2} + b = 5$$

$$\Rightarrow u_r = 2r + 4$$

b $$\sum_{r=n}^{2n} u_r = \sum_{r=1}^{2n} u_r - \sum_{r=1}^{n-1} u_r$$

$$= \left((2n)^2 + 5(2n)\right) - \left((n-1)^2 - 5(n-1)\right)$$

$$= 3n^2 + 7n + 4 = (n+1)(3n+4)$$

3 Substitute $w = x^2 + 1 \Rightarrow x^2 = w - 1 \Rightarrow x = \sqrt{w-1}$

So the equation becomes

$$\sqrt{w-1}(w-1) - 5(w-1) + 11\sqrt{w-1} - 15 = 0$$

$$\Rightarrow \sqrt{w-1}(w-1+11) = 5w + 10$$

Now squaring both sides

$$(w-1)(w+10)^2 = (5w+10)^2$$

$$\Rightarrow (w-1)(w^2 + 20w + 100) = 25w^2 + 100w + 100$$

$$\Rightarrow w^3 + 19w^2 + 80w - 100 = 25w^2 + 100w + 100$$

$$\Rightarrow w^3 - 6w^2 - 20w - 200 = 0$$