Volumes of revolution ME 5

1 \( V = \pi \int_{0}^{\pi} y^2 \, dx \)
\[
= \pi \int_{0}^{\pi} x^4 \left( 9 - x^2 \right) \, dx \\
= \pi \int_{0}^{\pi} \left( 9x^4 - x^6 \right) \, dx \\
= \pi \left[ \frac{9x^5}{5} - \frac{x^7}{7} \right]_{0}^{\pi} \\
= \pi \left( \frac{2187}{5} - \frac{2187}{7} \right) \\
= \frac{4374}{35} \pi
\]

2 a \( 2y^2 - 6\sqrt{x} + 3 = 0 \)
\[ y = 0 \implies 3 = 6\sqrt{x} \implies x = \frac{1}{4} \]

Curve cuts the \( x \)-axis when \( x = \frac{1}{4} \).

b \( V = \pi \int_{0.25}^{4} y^2 \, dx \)
\[
= \pi \int_{0.25}^{4} \left( 3\sqrt{x} - \frac{3}{2} \right) \, dx \\
= 3\pi \int_{0.25}^{4} \left( \sqrt{x} - \frac{1}{2} \right) \, dx \\
= 3\pi \left[ \frac{2}{3} x^{3/2} - \frac{1}{2} x \right]_{0.25}^{4} \\
= 3\pi \left( \frac{16}{3} - 2 - \left( \frac{1}{12} - \frac{1}{8} \right) \right) \\
= 3\pi \left( \frac{10}{3} + \frac{1}{24} \right) \\
= 3\pi \times \frac{81}{24} \\
= \frac{81}{8} \pi
\]

3 a \( f(x) = x^2 + 4x + 4 \)
\[ y = f(x) \Rightarrow y = x^2 + 4x + 4 \\
\]
\[ y = (x+2)^2 \\
\sqrt{y} = x + 2 \\
x = \sqrt{y} - 2 \\
x^2 = (\sqrt{y} - 2)^2 \\
x^2 = 4 - 4\sqrt{y} + y \]

b \( V = \pi \int_{4}^{9} x^3 \, dy \)
\[
= \pi \int_{4}^{9} \left( 4 - 4\sqrt{y} + y \right) \, dy \\
= \pi \left[ \frac{4y}{3} - \frac{y^{3/2}}{2} + \frac{y^2}{2} \right]_{4}^{9} \\
= \pi \left( \left( 36 - 72 + \frac{81}{2} \right) - \left( 16 - \frac{64}{3} + 8 \right) \right) \\
= \pi \left( \frac{9}{2} - \frac{8}{3} \right) \\
= \frac{11}{6} \pi
\]

4 a \( V = \pi \int_{0}^{1} y^2 \, dx \)
\[
= \pi \int_{0}^{1} \left( x^2 + 3 \right)^2 \, dx \\
= \pi \int_{0}^{1} \left( x^4 + 6x^2 + 9 \right) \, dx \\
= \pi \left[ \frac{1}{5} x^5 + 2x^3 + 9x \right]_{0}^{1} \\
= \frac{56}{5} \pi
\]
4 b  \(x = 1 \Rightarrow y = 4\) and \(x = 0 \Rightarrow y = 3\)

Volume generated by curve when rotated about the \(y\)-axis

\[
\pi \int_3^4 y^2 \, dy
\]

\[
= \pi \left[ \frac{1}{2} y^3 \right]_3^4
\]

\[
= \pi \left( 8 - 12 - \left( \frac{9}{2} - 9 \right) \right)
\]

\[
= \frac{1}{2} \pi
\]

Volume of cylinder generated when the line \(x = 1\) is rotated about the \(y\)-axis

\[= \pi \times 1 \times 4 = 4\pi\]

Volume obtained when region is rotated about the \(y\)-axis = \(4\pi - \frac{1}{2} \pi = \frac{7}{2} \pi\)

5 a Substituting \(x = 2\) and \(y = 4.5\) into the equation of the line gives

\[3 \times 2 + 4 \times 4.5 = 24\]

Substituting \(x = 2\) into the equation of the gives \(y = \frac{1}{4} \times 2 \times (2 + 1)^2 = 4.5\)

So \((2, 4.5)\) are the coordinates of the point of intersection \(A\).

5 b Volume \(V_1\) is generated by the curve between \(O\) and \(A\).

\[V_1 = \pi \int_0^1 y^2 \, dx\]

\[
= \frac{\pi}{16} \int_0^4 (x + 1)^2 \, dx
\]

\[
= \frac{\pi}{16} \int_0^4 x^2 \left( x^4 + 4x^3 + 6x^2 + 4x + 1 \right) \, dx
\]

\[
= \frac{\pi}{16} \left[ \left( \frac{128}{7} + \frac{128}{3} + \frac{192}{5} + 16 + \frac{8}{3} \right) \right]
\]

\[
= \frac{\pi}{16} \left( 240 + 560 + 504 + 210 + 35 \right)
\]

\[
= \frac{1549}{210} \pi
\]

Volume \(V_2\) generated by the line \(3x + 4y = 24\) is a cone with

\(r = 4.5\) and \(h = 6\)

\[V_2 = \frac{1}{3} \pi \times \left( \frac{9}{2} \right)^2 \times 6\]

\[= \frac{81}{2} \pi\]

Total volume = \(\frac{1549}{210} \pi + \frac{81}{2} \pi = \frac{5027}{105} \pi\)
6 a y-coordinate of points A and B
\[ 0.1 \times 2^2 + 4 = 4.4 \]

Volume of cylinder with \( r = 2 \) and \( h = 4.4 \)
\[ \pi \times 2^2 \times 4.4 = 17.6\pi \text{ cm}^3 \]

Volume generated by curve
\[ \pi \int_{0}^{4.4} y^2 \, dy \]
\[ = 10\pi \left[ \frac{1}{2} y^2 - 4y \right]_{0}^{4.4} \]
\[ = 10\pi \left( (9.68 - 17.6) - (8 - 16) \right) \]
\[ = 10\pi (-7.92 - (-8)) \]
\[ = 0.8\pi \]

Volume of bronze required
\[ 17.6\pi - 0.8\pi \]
\[ = 16.8\pi \text{ cm}^3 \]

b For example, the shape of the holder is unlikely to exactly follow the curve, or it doesn’t allow for any wastage.

7 Volume of the cap of the mushroom
\[ \pi \int_{0}^{2} x^2 \, dx \]
\[ = 4\pi \left[ 8\sqrt{y} - 4y \right]_{0}^{2} \]
\[ = 4\pi \left[ \frac{16}{3} y^{\frac{3}{2}} - 2y^2 \right]_{0}^{2} \]
\[ = 4\pi \left( \frac{128}{3} - 32 \right) \]
\[ = \frac{128}{3} \pi \]

The stem of mushroom is a cylinder with \( r = 1 \) and \( h = 4 \)

Volume of stem \[= \pi \times 1 \times 4 = 4\pi \]

Total volume of mushroom
\[ = \frac{128}{3} \pi + 4\pi = \frac{140}{3} \pi \text{ cm}^3 \]

8 The y-coordinate of the points of intersection of the two curves will be given by solving simultaneously
\[ 3y^2 + x^2 - 11y = 0 \quad (1) \]
\[ y = 2x^2 \quad (2) \]

\[ 3y^2 + \frac{y}{2} - 11y = 0 \]
\[ 6y^2 - 21y = 0 \]
\[ 2y^2 - 7y = 0 \]
\[ y(2y - 7) = 0 \]
\[ y = 0 \text{ or } y = \frac{7}{2} \]

Volume \( V_1 \) is generated by the curve
\[ 3y^2 + x^2 - 11y = 0 \] between \( y = 0 \) and \( y = \frac{7}{2} \)

\[ V_1 = \pi \int_{0}^{\frac{7}{2}} (11y - 3y^2) \, dy \]
\[ = \pi \left[ \frac{11}{2} y^2 - y^3 \right]_{0}^{\frac{7}{2}} \]
\[ = \pi \left( \frac{539}{8} - \frac{343}{8} \right) \]
\[ = \frac{49}{2} \pi \]

Volume \( V_2 \) is generated by the curve \[ y = 2x^2 \] between \( y = 0 \) and \( y = \frac{7}{2} \)

\[ V_2 = \pi \int_{0}^{\frac{7}{2}} \frac{y}{2} \, dy \]
\[ = \pi \left[ \frac{y^2}{4} \right]_{0}^{\frac{7}{2}} \]
\[ = \frac{49}{16} \pi \]

Volume generated by shaded region
\[ = V_1 - V_2 \]
\[ = \frac{49}{2} \pi - \frac{49}{16} \pi \]
\[ = \frac{343}{16} \pi \]
**Challenge**

a Using Pythagoras’ theorem:

\[ r^2 = x^2 + y^2 \]

So \[ y^2 = r^2 - x^2 \]

Area of disc = \[ \pi y^2 = \pi \left( r^2 - x^2 \right) \]

b Volume of sphere will be obtained by integrating this disc area between the limits

\[ x = -r \text{ to } x = r \]

\[ = \pi \int_{-r}^{r} \left( r^2 - x^2 \right) \, dx \]

\[ = \pi \left[ r^2x - \frac{1}{3}x^3 \right]_{-r}^{r} \]

\[ = \pi \left( \frac{2}{3}r^3 - \left( -\frac{2}{3}r^3 \right) \right) \]

\[ = \frac{4}{3} \pi r^3 \]