Volumes of revolution 5D

1 a k represents the height in metres of the tent at its centre, say $5 \leqslant k \leqslant 15$.

b $V = \pi\int_0^k x^2 \, dy$

$= 100\pi\int_0^k \left(k^2 - y^2\right) dy$

$= 100\pi\left[k^2 y - \frac{y^3}{3}\right]_0^k$

$= 100\pi\left(\frac{2}{3}k^3\right)$

$= \frac{200}{3}k^3\pi \text{ m}^3$

c The actual tent would probably not follow the exact same shape as the model. For example it would probably not be exactly circular looking down on it.

2 $y^2 = 4(16 - x)$

$x = 16 - \frac{y^2}{4}$

$V = \pi\int_{-8}^{8}\left(16 - \frac{y^2}{4}\right)^2 dy$

$= \pi\int_{-8}^{8}\left(256 - 8y^2 + \frac{y^4}{16}\right) dy$

$= \pi\left[256y - \frac{8y^3}{3} + \frac{y^5}{80}\right]_{-8}^{8}$

$= 2\pi\left(2048 - \frac{4096}{3} + \frac{32\,768}{80}\right)$

$= 4096\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$

$= 4096\pi\left(\frac{8}{15}\right)$

$= \frac{32\,768}{15}\pi \text{ cm}^3$

3 a $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow y^2 = 4\left(1 - \frac{x^2}{9}\right)$

$V = 4\pi\int_{-3}^{3}\left(1 - \frac{x^2}{9}\right) dx$

$= 4\pi\left[x - \frac{x^3}{27}\right]_{-3}^{3}$

$= 4\pi\left((3-1) - (-3+1)\right)$

$= 16\pi \text{ cm}^3$

b $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow x^2 = 9\left(1 - \frac{y^2}{4}\right)$

$V = 9\pi\int_{-2}^{2}\left(1 - \frac{y^2}{4}\right) dy$

$= 9\pi\left[y - \frac{y^3}{12}\right]_{-2}^{2}$

$= 9\pi\left(\left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)\right)$

$= 24\pi \text{ cm}^3$

c The solid formed by rotating the ellipse about the x-axis will be more like the shape of an egg. The solid formed by rotating about the y-axis will have a flatter disc shape.

4 Volume of sand which flows in 5 minutes is 40 cm^3.

Let b be the required height of the sand.

$40 = \pi\int_0^b \left(\sqrt[3]{y}\right)^2 dy$

$40 = \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^b$

$40 = \frac{3}{5}\pi b^{\frac{5}{3}}$

$b^{\frac{5}{3}} = \frac{200}{3\pi}$

$b = \left(\frac{200}{3\pi}\right)^{\frac{3}{5}} = 6.25 \text{ cm}$

5 a $y = 0.02x^3$

$y = 18 \Rightarrow x^3 = 900$

Diameter of bowl $= 2 \times \sqrt[3]{900} = 19.3$ cm

b $V = \pi \int_0^{18} x^2 \, dy$

$= \pi \int_0^{18} \left(\sqrt[3]{50y}\right)^2 \, dy$

$= \sqrt[3]{2500}\pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^{18}$

$= \sqrt[3]{2500}\pi \left(\frac{3}{5} 18^{\frac{5}{3}}\right)$

$= 3162.8 \text{ cm}^3$

c Area of $R = \int_0^{12} \sqrt[3]{50y} \, dy$

$= \sqrt[3]{50} \times \left[\frac{3}{4} y^{\frac{4}{3}}\right]_0^{12}$

$= 75.9 \text{ cm}^2$

d Let V_p be the volume of liquid mixed by the paddle.

$V_p = \pi \int_0^{12} \left(\sqrt[3]{50y}\right)^2 \, dy$

$= \sqrt[3]{2500}\pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^{12}$

$= \sqrt[3]{2500}\pi \left(\frac{3}{5} 12^{\frac{5}{3}}\right)$

$= 1609.1 \text{ cm}^3$

Proportion of volume in bowl that can be mixed $= \frac{1609.1}{3162.8} = 0.509$

Proportion could be found directly:

$\left(\frac{12}{18}\right)^{\frac{5}{3}} = \left(\frac{2}{3}\right)^{\frac{5}{3}} = 0.509$

6 a $V = \pi \int_0^{20} x^2 \, dy$

$= \pi \int_0^{20} \left(5 - \sqrt{y}\right)^2 \, dy$

$= \pi \int_0^{20} \left(25 - 10\sqrt{y} + y\right) \, dy$

$= \pi \left[25y - \frac{20}{3} y^{\frac{3}{2}} + \frac{1}{2} y^2\right]_0^{20}$

$= \pi \left(500 - \frac{20}{3} \times 20^{\frac{3}{2}} + 200\right)$

$= 325.8 \text{ cm}^3$

b Volume of water to height 10 cm

$= \pi \left[25y - \frac{20}{3} y^{\frac{3}{2}} + \frac{1}{2} y^2\right]_0^{10}$

$= \pi \left(250 - \frac{20}{3} \times 10^{\frac{3}{2}} + 50\right)$

$= 280.2 \text{ cm}^3$

So adding another 50 cm^3 will take the volume over the 325.8 cm^3 capacity of the vase and it will therefore overflow.

7 $y = 2x + 18$ intersects the x-axis at (−9, 0) and the y-axis at (0, 18) so the uppermost part of the top is a cone with $r = 9$ and $h = 18$.

Volume of this cone $= \frac{1}{3}\pi \times 81 \times 18$
$= 486\pi \text{ cm}^3$

The curve intersects the y-axis at (0, −6) so the volume of the lower part of the top is

$$\pi\int_{-6}^{0} x^2 \, dy = \pi\int_{-6}^{0}\left(\frac{y^2}{4} - 9\right)^2 \, dy$$
$$= \pi\int_{-6}^{0}\left(\frac{y^4}{16} - \frac{9y^2}{2} + 81\right) dy$$
$$= \pi\left[\frac{1}{80}y^5 - \frac{3}{2}y^3 + 81y\right]_{-6}^{0}$$
$$= \pi\left((0) - \left(-\frac{486}{5} + 324 - 486\right)\right)$$
$$= \pi\left(\frac{486 - 1620 + 2430}{5}\right)$$
$$= \frac{1296}{5}\pi$$
$$= 259.2\pi \text{ cm}^3$$

Total volume of top $= 486\pi + 259.2\pi$
$= 745.2\pi \text{ cm}^3$

Challenge

Volume of frustum generated by the line $3y + 4x = 24$

$$= \pi\int_0^4 x^2 \, dy$$
$$= \pi\int_0^4 \left(6 - \frac{3y}{4}\right)^2 \, dy$$
$$= \pi\int_0^4 \left(36 - 9y + \frac{9y^2}{16}\right) dy$$
$$= \pi\left[36y - \frac{9}{2}y^2 + \frac{3}{16}y^3\right]_0^4$$
$$= \pi(144 - 72 + 12)$$
$$= 84\pi \text{ cm}^3$$

Volume removed from top of frustum by rotating the curve $y = \frac{1}{9}x^2 + 3$ about the y-axis

$$= \pi\int_3^4 x^2 \, dy$$
$$= \pi\int_3^4 (9y - 27) \, dy$$
$$= \pi\left[\frac{9}{2}y^2 - 27y\right]_3^4$$
$$= \pi\left((72 - 108) - \left(\frac{81}{2} - 81\right)\right)$$
$$= \pi\left(-36 - \frac{81}{2} + 81\right)$$
$$= \frac{9}{2}\pi \text{ cm}^3$$

Volume of place-holder

$$= 84\pi - \frac{9}{2}\pi$$
$$= \frac{159}{2}\pi \text{ cm}^3$$