Volumes of revolution 5C

1 a $3x+2y=27$

When $y=0,\ x=9$

$y=\frac{1}{2}(27-3x)$

$y^2=\frac{1}{4}(729-162x+9x^2)$

$V=\frac{\pi}{4}\int_0^9(729-162x+9x^2)\,\mathrm{d}x$

$=\frac{\pi}{4}\left[729x-81x^2+3x^3\right]_0^9$

$=\frac{2187}{4}\pi$

b $3x+2y=27$

When $x=0,\ y=13.5$

$x=\frac{1}{3}(27-2y)$

$x^2=\frac{1}{9}(729-108y+4y^2)$

$V=\frac{\pi}{9}\int_0^{13.5}(729-108y+4y^2)\,\mathrm{d}y$

$=\frac{\pi}{9}\left[729y-54y^2+\frac{4y^3}{3}\right]_0^{13.5}$

$=\frac{729}{2}\pi$

c Rotation about the *x*-axis: $r=13.5,\ h=9$

$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\times13.5^2\times9=\frac{2187}{4}\pi$

Rotation about the *y*-axis: $r=9,\ h=13.5$

$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\times9^2\times13.5=\frac{729}{2}\pi$

2 a Substituting $x=2$ into the equation of the line gives $y=8$.

Substituting $x=2$ into the equation of the curve gives $y=\frac{1}{2}\times4\times(2+2)=8$.

So (2, 8) are the coordinates of the point of intersection *A*.

b Volume V_1 is generated by rotating the curve about the *x*-axis.

$V_1=\pi\int_0^2\left(\frac{1}{2}x^2(x+2)\right)^2\,\mathrm{d}x$

$=\frac{\pi}{4}\int_0^2x^4(x^2+4x+4)\,\mathrm{d}x$

$=\frac{\pi}{4}\int_0^2(x^6+4x^5+4x^4)\,\mathrm{d}x$

$=\frac{\pi}{4}\left[\frac{x^7}{7}+\frac{2x^6}{3}+\frac{4x^5}{5}\right]_0^2$

$=\frac{\pi}{4}\left(\frac{128}{7}+\frac{128}{3}+\frac{128}{5}\right)$

$=\pi\left(\frac{32}{7}+\frac{32}{3}+\frac{32}{5}\right)$

$=\frac{480+1120+672}{105}\pi$

$=\frac{2272}{105}\pi$

Volume V_2 is generated by rotating the line about the *x*-axis, so is a cone.

When $y=0$, $x=4$ so $r=8,\ h=2$

$V_2=\frac{1}{3}\pi\times64\times2=\frac{128}{3}\pi$

Total volume $=\frac{2272}{105}\pi+\frac{128}{3}\pi$

$=\frac{2272+4480}{105}\pi$

$=\frac{6752}{105}\pi$

3 a Substituting $x=2$ into the equation of the line gives $y=4$.

Substituting $x=2$ into the equation of the curve gives $y=-\frac{1}{2}\times4\times(2-4)=4$.

So (2, 4) are the coordinates of the point of intersection *A*.

$y=0\Rightarrow x=4$

so the coordinates of *B* are (4, 0).

3 b Volume generated by area under curve is given by $\pi\int_2^4\left(-\frac{1}{2}x^2(x-4)\right)^2 \mathrm{d}x$

$=\frac{\pi}{4}\int_2^4 x^4\left(x^2-8x+16\right)\mathrm{d}x$

$=\frac{\pi}{4}\int_2^4\left(x^6-8x^5+16x^4\right)\mathrm{d}x$

$=\frac{\pi}{4}\left[\frac{x^7}{7}-\frac{4x^6}{3}+\frac{16x^5}{5}\right]_2^4$

$=\frac{\pi}{4}\left(\frac{16\,384}{105}-\frac{3712}{105}\right)$

$=\frac{3168}{105}\pi$

Volume generated by line is a cone with $r=4$ and $h=2$.

Volume of cone $=\frac{1}{3}\pi\times16\times2=\frac{32}{3}\pi$

Volume generated by R $=\frac{3168}{105}\pi-\frac{32}{3}\pi$

$=\frac{2048}{105}\pi$

4 a Substituting $x=-2$ into the linear equation $2x-y=-6$ gives $y=2$.

Substituting $x=-2$ into the equation of the curve $y=\frac{1}{2}x^2$ gives $y=2$.

So (−2, 2) are the coordinates of the point of intersection *A*.

Substituting $x=2$ into the linear equation $2x+y=6$ gives $y=2$.

Substituting $x=2$ into the equation of the curve $y=\frac{1}{2}x^2$ gives $y=2$.

So (2, 2) are the coordinates of the point of intersection *B*.

4 b Volume generated by the curve

$=\pi\int_0^2 x^2\,\mathrm{d}y$

$=\pi\int_0^2 2y\,\mathrm{d}y$

$=\pi\left[y^2\right]_0^2$

$=4\pi$

Volume generated by the line $2x+y=6$ is a cone with $r=2$ and $h=4$

Volume of cone $=\frac{1}{3}\pi\times4\times4=\frac{16}{3}\pi$

Total volume $=4\pi+\frac{16}{3}\pi=\frac{28}{3}\pi$

5 Volume generated by *C* about the *y*-axis

$=\pi\int_0^2\left(4-y^2\right)\mathrm{d}y$

$=\pi\left[4y-\frac{y^3}{3}\right]_0^2$

$=\pi\left(8-\frac{8}{3}\right)=\frac{16}{3}\pi$

Volume of cylinder with $r=3$ and $h=3$
$=\pi\times9\times3=27\pi$

Volume generated by *R*

$=27\pi-\frac{16}{3}\pi$

$=\frac{65}{3}\pi$

6 $y = -\frac{1}{5}x^2 + 5$ intersects the y-axis at (0, 5)

Rearranging… $x^2 = 25 - 5y$

Volume generated by the curve about the y-axis

$$= \pi\int_0^5 x^2 \, dy$$
$$= \pi\int_0^5 (25 - 5y) \, dy$$
$$= \pi\left[25y - \frac{5y^2}{2}\right]_0^5$$
$$= \pi\left(125 - \frac{125}{2}\right)$$
$$= \frac{125}{2}\pi$$

Volume to be subtracted is a cone with $r = 4$ and $h = 4$.

Volume of cone $= \frac{1}{3}\pi \times 16 \times 4 = \frac{64}{3}\pi$

Volume generated by $R = \frac{125}{2}\pi - \frac{64}{3}\pi$
$$= \frac{247}{6}\pi$$

7 $6y^2 - x^3 + 4x = 0$

$y = 0 \Rightarrow 4x - x^3 = 0$

$x(4 - x^2) = 0$

$x = 2$ since $x > 0$

So C cuts the x-axis at (2, 0).

Volume generated by the curve about the x-axis

$$= \pi\int_2^4 y^2 \, dx$$
$$= \frac{\pi}{6}\int_2^4 (x^3 - 4x) \, dx$$
$$= \frac{\pi}{6}\left[\frac{x^4}{4} - 2x^2\right]_2^4$$
$$= \frac{\pi}{6}((64 - 32) - (4 - 8))$$
$$= \frac{36\pi}{6}$$
$$= 6\pi$$

Volume generated by lines is a cone with $r = 4$ and $h = 3$

Volume of cone $= \frac{1}{3}\pi \times 16 \times 3 = 16\pi$

Required volume $= 16\pi - 6\pi = 10\pi$

8 Volume generated by rotating the curve $y = 4 - x^2$ about the x-axis

$$= \pi\int_0^1 \left(4 - x^2\right)^2 \, dx$$

$$= \pi\int_0^1 \left(16 - 8x^2 + x^4\right) \, dx$$

$$= \pi\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= \pi\left(16 - \frac{8}{3} + \frac{1}{5}\right)$$

$$= \frac{203}{15}\pi$$

Volume generated by rotating the curve $y = \sqrt[3]{x}$ about the x-axis

$$= \pi\int_0^1 x^{\frac{2}{3}} \, dx$$

$$= \pi\left[\frac{3x^{\frac{5}{3}}}{5}\right]_0^1$$

$$= \frac{3}{5}\pi$$

Required volume $= \frac{203}{15}\pi - \frac{3}{5}\pi = \frac{194}{15}\pi$

9 a $x^2 + y^2 = 11$

$$x^2 + \left(x^2 + 1\right)^2 = 11$$

$$x^2 + x^4 + 2x^2 + 1 = 11$$

$$x^4 + 3x^2 - 10 = 0$$

$$\left(x^2 + 5\right)\left(x^2 - 2\right) = 0$$

$$x = \pm\sqrt{2}$$

So the x coordinates of the points of intersection are $-\sqrt{2}$ and $\sqrt{2}$.

9 b Volume generated by $x^2 + y^2 = 11$ between $x = \pm\sqrt{2}$ is $\pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(11 - x^2\right) \, dx$

$$= \pi\left[11x - \frac{x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{\pi}{3}\left(\left(33\sqrt{2} - 2\sqrt{2}\right) - \left(-33\sqrt{2} + 2\sqrt{2}\right)\right)$$

$$= \frac{62\sqrt{2}}{3}\pi$$

Volume generated by $y = x^2 + 1$ between $x = \pm\sqrt{2}$ is $\pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(x^2 + 1\right)^2 \, dx$

$$= \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(x^4 + 2x^2 + 1\right) \, dx$$

$$= \pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \pi\left(\frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3} + \sqrt{2}\right) - \left(-\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} - \sqrt{2}\right)$$

$$= \pi\left(\frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{3} + 2\sqrt{2}\right)$$

$$= \frac{94\sqrt{2}}{15}\pi$$

Required volume

$$= \frac{62\sqrt{2}}{3}\pi - \frac{94\sqrt{2}}{15}\pi$$

$$= \frac{72\sqrt{2}}{5}\pi$$

= 63.98 (2 d.p.)

Challenge

Required volume V will be the volume V_1 of the cone with $r = 5$ and $h = \frac{40}{3}$

minus the volume V_2 of the cone with $r = 2$ and $h = \frac{40}{3} - 8 = \frac{16}{3}$

minus the volume V_3 generated by the curve $y = \frac{64}{x^3}$ between $y = 1$ and $y = 8$.

$$V_1 = \frac{1}{3}\pi \times 25 \times \frac{40}{3} = \frac{1000}{9}\pi$$

$$V_2 = \frac{1}{3}\pi \times 4 \times \frac{16}{3} = \frac{64}{9}\pi$$

$$\begin{aligned} V_3 &= \pi\int_1^8 x^2 \, \mathrm{d}y \\ &= \pi\int_1^8 \left(\sqrt[3]{\frac{64}{y}}\right)^2 \mathrm{d}y \\ &= 16\pi\int_1^8 y^{-\frac{2}{3}} \, \mathrm{d}y \\ &= 48\pi\left[y^{\frac{1}{3}}\right]_1^8 \\ &= 48\pi \end{aligned}$$

$$\begin{aligned} V &= V_1 - V_2 - V_3 \\ &= \frac{1000}{9}\pi - \frac{64}{9}\pi - 48\pi \\ &= 56\pi \end{aligned}$$