Volumes of revolution 5B

1 a $V = \pi\int_2^5 \left(\frac{1}{2}y + 1\right)^2 \, dy$

$= \pi\int_2^5 \left(\frac{1}{4}y^2 + y + 1\right) dy$

$= \pi\left[\frac{y^3}{12} + \frac{y^2}{2} + y\right]_2^5$

$= \pi\left(\left(\frac{125}{12} + \frac{25}{2} + 5\right) - \left(\frac{8}{12} + \frac{4}{2} + 2\right)\right)$

$= \pi\left(\frac{335}{12} - \frac{14}{3}\right)$

$= \frac{279}{12}\pi$

$= \frac{93}{4}\pi$

b $y = 2\sqrt{x} \Rightarrow x = \frac{y^2}{4}$

$V = \pi\int_0^1 \left(\frac{y^2}{4}\right)^2 \, dy$

$= \pi\int_0^1 \frac{y^4}{16} \, dy$

$= \pi\left[\frac{y^5}{80}\right]_0^1$

$= \frac{1}{80}\pi$

c $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

$V = \pi\int_1^3 \left(\frac{1}{y}\right)^2 \, dy$

$= \pi\int_1^3 \frac{1}{y^2} \, dy$

$= \pi\left[-\frac{1}{y}\right]_1^3$

$= \pi\left(-\frac{1}{3} + 1\right)$

$= \frac{2}{3}\pi$

1 d $y = 2x^2 - 4 \Rightarrow x^2 = \frac{1}{2}(y + 4)$

$x = 11 \Rightarrow y = 238$

$x = 5 \Rightarrow y = 46$

$V = \pi\int_{46}^{238} \frac{1}{2}(y + 4) \, dy$

$= \frac{\pi}{2}\left[\frac{y^2}{2} + 4y\right]_{46}^{238}$

$= \frac{\pi}{2}((28\,322 + 952) - (1058 + 184))$

$= \frac{\pi}{2}(29\,274 - 1242)$

$= 14\,016\pi$

2 $V = \pi\int_1^4 \left(\frac{1}{2}y^2 + 1\right)^2 \, dy$

$= \pi\int_1^4 \left(\frac{y^4}{4} + y^2 + 1\right) dy$

$= \pi\left[\frac{y^5}{20} + \frac{y^3}{3} + y\right]_1^4$

$= \pi\left(\left(\frac{1024}{20} + \frac{64}{3} + 4\right) - \left(\frac{1}{20} + \frac{1}{3} + 1\right)\right)$

$= \pi\left(\frac{3072 + 1280 + 240 - 3 - 20 - 60}{60}\right)$

$= \frac{1503}{20}\pi$

3 a $A = \int_4^9 \left(\sqrt{y} + \frac{1}{y^2}\right) dy$

$= \left[\frac{2}{3}y^{\frac{3}{2}} - \frac{1}{y}\right]_4^9$

$= \left(18 - \frac{1}{9}\right) - \left(\frac{16}{3} - \frac{1}{4}\right)$

$= \frac{648 - 4 - 192 + 9}{36}$

$= \frac{461}{36}$

3 b $V = \pi\int_4^9 \left(\sqrt{y} + \frac{1}{y^2}\right)^2 \text{ d}y$

$= \pi\int_4^9 \left(y + 2y^{-\frac{3}{2}} + y^{-4}\right) \text{d}y$

$= \pi\left[\frac{y^2}{2} - 4y^{-\frac{1}{2}} - \frac{y^{-3}}{3}\right]_4^9$

$= \pi(39.166 - 5.995)$

$= 104.21$ (2 d.p.)

4 a $A = \int_1^4 \left(y^2 - 6y + 10\right) \text{d}y$

$= \left[\frac{y^3}{3} - 3y^2 + 10y\right]_1^4$

$= \left(\left(\frac{64}{3} - 48 + 40\right) - \left(\frac{1}{3} - 3 + 10\right)\right)$

$= 6$

b $V = \pi\int_1^4 \left(y^2 - 6y + 10\right)^2 \text{ d}y$

$= \pi\int_1^4 \left(y^4 - 12y^3 + 56y^2 - 120y + 100\right) \text{d}y$

$= \pi\left[\frac{y^5}{5} - 3y^4 + \frac{56y^3}{3} - 60y^2 + 100y\right]_1^4$

$= \pi\left(\left(\frac{1024}{5} + \frac{3584}{3} - 1328\right) - \left(\frac{1}{5} + \frac{56}{3} + 37\right)\right)$

$= \frac{78}{5}\pi$

5 $y = 2x^2 + 5$

When $x = 0, y = 5$

So C cuts the y-axis at (0, 5)

Rearranging $x^2 = \frac{1}{2}(y - 5)$

$V = \pi\int_5^{10} \frac{1}{2}(y - 5) \text{ d}y$

$= \frac{\pi}{2}\left[\frac{y^2}{2} - 5y\right]_5^{10}$

$= \frac{\pi}{2}\left((50 - 50) - \left(\frac{25}{2} - 25\right)\right)$

$= \frac{25}{4}\pi$

6 a $y = x^2 - 2x + 1$

$y = (x - 1)^2$

$\sqrt{y} = x - 1 \quad y > 0$

$x = \sqrt{y} + 1$

$x^2 = \left(\sqrt{y} + 1\right)^2$

$x^2 = y + 2\sqrt{y} + 1$

b $V = \pi\int_1^9 \left(y + 2\sqrt{y} + 1\right) \text{d}y$

$= \pi\left[\frac{y^2}{2} + \frac{4}{3}y^{\frac{3}{2}} + y\right]_1^9$

$= \pi\left(\left(\frac{81}{2} + 36 + 9\right) - \left(\frac{1}{2} + \frac{4}{3} + 1\right)\right)$

$= \frac{248}{3}\pi$

7 $y^3 + x^2 - 2y = 4$

So $x^2 = 4 + 2y - y^3$

$V = \pi\int_0^2 \left(4 + 2y - y^3\right) \text{d}y$

$= \pi\left[4y + y^2 - \frac{y^4}{4}\right]_0^2$

$= \pi((8 + 4 - 4) - 0)$

$= 8\pi$

8 $y^2 = \dfrac{1}{2x+1}$

So $2x+1 = \dfrac{1}{y^2}$

$$x = \frac{1}{2}\left(\frac{1}{y^2} - 1\right)$$

$$x^2 = \frac{1}{4}\left(\frac{1}{y^4} - \frac{2}{y^2} + 1\right)$$

$$V = \frac{\pi}{4}\int_1^4 \left(\frac{1}{y^4} - \frac{2}{y^2} + 1\right) \mathrm{d}y$$

$$= \frac{\pi}{4}\left[-\frac{1}{3y^3} + \frac{2}{y} + y\right]_1^4$$

$$= \frac{\pi}{4}\left(\left(-\frac{1}{192} + \frac{1}{2} + 4\right) - \left(-\frac{1}{3} + 2 + 1\right)\right)$$

$$= \frac{\pi}{4}\left(\frac{-1+96+768+64-384-192}{192}\right)$$

$$= \frac{117}{256}\pi$$

$$= 1.44 \text{ (2 d.p.)}$$

9 Gradient of hypotenuse $= \dfrac{h}{r}$

So equation of hypotenuse is $y = \dfrac{h}{r}x$

So $x^2 = \dfrac{r^2}{h^2}y^2$

$$V = \pi\int_0^h \frac{r^2}{h^2}y^2 \ \mathrm{d}y$$

$$= \frac{\pi r^2}{h^2}\left[\frac{y^3}{3}\right]_0^h$$

$$= \frac{\pi r^2}{h^2}\left(\frac{h^3}{3} - 0\right)$$

$$= \frac{1}{3}\pi r^2 h$$