

Series 3B

$$1 \text{ a } \sum_{r=1}^4 r^2 = \frac{1}{6} \times 4 \times 5 \times 9 = 30$$

$$b \sum_{r=1}^{40} r^2 = \frac{1}{6} \times 40 \times 41 \times 81 = 22\,140$$

$$c \sum_{r=21}^{40} r^2 = \sum_{r=1}^{40} r^2 - \sum_{r=1}^{20} r^2 \\ = \frac{1}{6} \times 40 \times 41 \times 81 - \frac{1}{6} \times 20 \times 21 \times 41 \\ = 19\,270$$

$$d \sum_{r=1}^{99} r^3 = \frac{1}{4} \times 99^2 \times 100^2 = 24\,502\,500$$

$$e \sum_{r=1}^{100} r^3 = \frac{1}{4} \times 100^2 \times 101^2 = 25\,502\,500$$

$$f \sum_{r=100}^{200} r^3 = \sum_{r=1}^{200} r^3 - \sum_{r=1}^{99} r^3 \\ = \frac{1}{4} \times 200^2 \times 201^2 - \frac{1}{4} \times 99^2 \times 100^2 \\ = 379\,507\,500$$

$$g \sum_{r=1}^k r^2 + \sum_{r=k+1}^{80} r^2 = \sum_{r=1}^{80} r^2 \\ = \frac{1}{6} \times 80 \times 81 \times 161 \\ = 173\,880$$

$$2 \text{ a } \sum_{r=1}^{2n} r^2 = \frac{1}{6} (2n)(2n+1)(4n+1) \\ = \frac{1}{3} n(2n+1)(4n+1)$$

$$b \sum_{r=1}^{2n-1} r^2 = \frac{1}{6} (2n-1)(2n)(2(2n-1)+1) \\ = \frac{1}{6} \times 2n(2n-1)(4n-1) \\ = \frac{1}{3} n(2n-1)(4n-1)$$

$$2 \text{ c } \sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2 \\ = \frac{1}{3} n(2n+1)(4n+1) - \frac{1}{6} (n-1)(n)(2(n-1)+1) \\ = \frac{1}{6} n(2(2n+1)(4n+1) - (n-1)(2n-1)) \\ = \frac{1}{6} n(16n^2 + 12n + 2 - 2n^2 + 3n - 1) \\ = \frac{1}{6} n(14n^2 + 15n + 1) \\ = \frac{1}{6} n(n+1)(14n+1)$$

$$3 \text{ Let } n+k=m \\ \Rightarrow \sum_{r=1}^m r^3 = \frac{1}{4} m^2 (m+1)^2 \\ \text{and so } \sum_{r=1}^{n+k} r^3 = \frac{1}{4} (n+k)^2 (n+k+1)^2$$

$$4 \text{ a } \sum_{r=n+1}^{3n} r^3 = \sum_{r=1}^{3n} r^3 - \sum_{r=1}^n r^3 \\ = \frac{1}{4} (3n)^2 (3n+1)^2 - \frac{1}{4} n^2 (n+1)^2 \\ = \frac{1}{4} n^2 (9(3n+1)^2 - (n+1)^2) \\ = \frac{1}{4} n^2 (81n^2 - n^2 + 54n - 2n + 9 - 1) \\ = n^2 (20n^2 + 13n + 2) \\ = n^2 (4n+1)(5n+2)$$

$$b \sum_{r=11}^{30} r^3 = 10^2 \times 41 \times 52 = 213\,200$$

$$5 \text{ a } \sum_{r=n}^{2n} r^3 = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n-1} r^3 \\ = \frac{1}{4} (2n)^2 (2n+1)^2 - \frac{1}{4} (n-1)^2 n^2 \\ = \frac{1}{4} n^2 (4(2n+1)^2 - (n-1)^2) \\ = \frac{1}{4} n^2 (16n^2 + 16n + 4 - n^2 + 2n - 1) \\ = \frac{1}{4} n^2 (15n^2 + 18n + 3) \\ = \frac{3}{4} n^2 (5n^2 + 6n + 1) \\ = \frac{3}{4} n^2 (n+1)(5n+1)$$

$$5 \text{ b } \sum_{r=30}^{60} r^3 = \frac{3}{4} \times 30^2 \times 31 \times 151 = 3\,159\,675$$

$$6 \text{ a } \sum_{m=1}^{30} (m^2 - 1) = \sum_{m=1}^{30} m^2 - \sum_{m=1}^{30} 1 \\ = \sum_{m=1}^{30} m^2 - 30 \\ = \frac{1}{6} \times 30 \times 31 \times 61 - 30 = 9425$$

$$b \sum_{r=1}^{40} r(r+4) = \sum_{r=1}^{40} r^2 + 4r = \sum_{r=1}^{40} r^2 + 4 \sum_{r=1}^{40} r \\ = \frac{1}{6} \times 40 \times 41 \times 81 + 4 \times \frac{1}{2} \times 40 \times 41 \\ = 22\,140 + 3280 = 25\,420$$

$$c \sum_{r=1}^{80} r(r^2 + 3) = \sum_{r=1}^{80} (r^3 + 3r) = \sum_{r=1}^{80} r^3 + 3 \sum_{r=1}^{80} r \\ = \frac{1}{4} \times 80^2 \times 81^2 + 3 \times \frac{1}{2} \times 80 \times 81 \\ = 10\,497\,600 + 9720 = 10\,507\,320$$

$$d \sum_{r=11}^{35} (r^3 - 2) = \sum_{r=1}^{35} (r^3 - 2) - \sum_{r=1}^{10} (r^3 - 2) \\ = \sum_{r=1}^{35} r^3 - \sum_{r=1}^{10} r^3 - 2 \sum_{r=1}^{35} 1 - 2 \sum_{r=1}^{10} 1 \\ = \frac{1}{4} \times 35^2 \times 36^2 - \frac{1}{4} \times 10^2 \times 11^2 - 2(35 - 10) \\ = 396\,900 - 3025 - 50 = 393\,825$$

$$7 \text{ a } \sum_{r=1}^n (r+2)(r+5) = \sum_{r=1}^n (r^2 + 7r + 10) \\ = \sum_{r=1}^n r^2 + 7 \sum_{r=1}^n r + 10 \sum_{r=1}^n 1 \\ = \frac{1}{6} n(n+1)(2n+1) + \frac{7}{2} n(n+1) + 10n \\ = \frac{1}{6} n(2n^2 + 3n + 1 + 21n + 21 + 60) \\ = \frac{1}{3} n(n^2 + 12n + 41)$$

$$7 \text{ b } \sum_{r=10}^{50} (r+2)(r+5) \\ = \sum_{r=1}^{50} (r+2)(r+5) - \sum_{r=1}^9 (r+2)(r+5) \\ = \frac{1}{3} \times 50 \times (50^2 + 600 + 41) \\ \quad - \frac{1}{3} \times 9 \times (9^2 + 108 + 41) \\ = 52\,350 - 690 = 51\,660$$

$$8 \text{ a } \sum_{r=1}^n (r^2 + 3r + 1) = \sum_{r=1}^n r^2 + 3 \sum_{r=1}^n r + \sum_{r=1}^n 1 \\ = \frac{1}{6} n(n+1)(2n+1) + \frac{3}{2} n(n+1) + n \\ = \frac{1}{6} n(2n^2 + 3n + 1 + 9n + 9 + 6) \\ = \frac{1}{3} n(n^2 + 6n + 8) = \frac{1}{3} n(n+2)(n+4) \\ a = 2, b = 4$$

$$b \sum_{r=19}^{40} (r^2 + 3r + 1) \\ = \sum_{r=1}^{40} (r^2 + 3r + 1) - \sum_{r=1}^{18} (r^2 + 3r + 1) \\ = \frac{1}{3} \times 40 \times 42 \times 44 - \frac{1}{3} \times 18 \times 20 \times 22 \\ = 22\,000$$

$$9 \text{ a } \sum_{r=1}^n r^2(r-1) = \sum_{r=1}^n r^3 - \sum_{r=1}^n r^2 \\ = \frac{1}{4} n^2(n+1)^2 - \frac{1}{6} n(n+1)(2n+1) \\ = \frac{1}{12} n(n+1)(3n(n+1) - 2(2n+1)) \\ = \frac{1}{12} n(n+1)(3n^2 - n - 2)$$

$$b \sum_{r=1}^{2n-1} r^2(r-1) \\ = \frac{1}{12} (2n-1)((2n-1)+1)(3(2n-1)^2 \\ \quad - (2n-1) - 2) \\ = \frac{1}{12} \times 2n(2n-1)(12n^2 - 14n + 2) \\ = \frac{1}{3} n(2n-1)(6n^2 - 7n + 1)$$

$$\begin{aligned}
 \mathbf{10\ a} \quad \sum_{r=1}^n (r+1)(r+3) &= \sum_{r=1}^n (r^2 + 4r + 3) \\
 &= \sum_{r=1}^n r^2 + 4 \sum_{r=1}^n r + 3 \sum_{r=1}^n 1 \\
 &= \frac{1}{6}n(n+1)(2n+1) + 2n(n+1) + 3n \\
 &= \frac{1}{6}n(2n^2 + 3n + 1 + 12n + 12 + 18) \\
 &= \frac{1}{6}n(2n^2 + 15n + 31)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \sum_{r=n+1}^{2n} (r+1)(r+3) \\
 &= \sum_{r=1}^{2n} (r+1)(r+3) - \sum_{r=1}^n (r+1)(r+3) \\
 &= \frac{1}{6} \times 2n(8n^2 + 30n + 31) \\
 &\quad - \frac{1}{6} \times n \times (2n^2 + 15n + 31) \\
 &= \frac{1}{6}n(16n^2 + 60n + 62 - 2n^2 - 15n - 31) \\
 &= \frac{1}{6}n(14n^2 + 45n + 31)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{11\ a} \quad \sum_{r=1}^n (r+3)(r+4) &= \sum_{r=1}^n r^2 + 7r + 12 \\
 &= \sum_{r=1}^n r^2 + 7 \sum_{r=1}^n r + 12 \sum_{r=1}^n 1 \\
 &= \frac{1}{6}n(n+1)(2n+1) + \frac{7}{2}n(n+1) + 12n \\
 &= \frac{1}{6}n(2n^2 + 3n + 1 + 21n + 21 + 72) \\
 &= \frac{1}{3}n(n^2 + 12n + 47)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \sum_{r=n+1}^{3n} (r+3)(r+4) \\
 &= \sum_{r=1}^{3n} (r+3)(r+4) - \sum_{r=1}^n (r+3)(r+4) \\
 &= \frac{1}{3} \times 3n(9n^2 + 36n + 47) \\
 &\quad - \frac{1}{3}n(n^2 + 12n + 47) \\
 &= \frac{1}{3}n(27n^2 + 108n + 141 - n^2 - 12n - 47) \\
 &= \frac{1}{3}n(26n^2 + 96n + 94) \\
 &= \frac{2}{3}n(13n^2 + 48n + 47)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{12\ a} \quad \sum_{r=1}^n r(r+3)^2 &= \sum_{r=1}^n r^3 + 6r^2 + 9r \\
 &= \sum_{r=1}^n r^3 + 6 \sum_{r=1}^n r^2 + 9 \sum_{r=1}^n r \\
 &= \frac{1}{4}n^2(n+1)^2 + n(n+1)(2n+1) + \frac{9}{2}n(n+1) \\
 &= \frac{1}{4}n(n+1)(n(n+1) + 4(2n+1) + 18) \\
 &= \frac{1}{4}n(n+1)(n^2 + 9n + 22)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \sum_{r=10}^{20} r(r+3)^2 \\
 &= \sum_{r=1}^{20} r(r+3)^2 - \sum_{r=1}^9 r(r+3)^2 \\
 &= \frac{1}{4} \times 20 \times 21 \times (20^2 + 180 + 22) \\
 &\quad - \frac{1}{4} \times 9 \times 10 \times (9^2 + 81 + 22) \\
 &= 59\,070
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{13\ a} \quad \sum_{r=1}^{kn} (2r-1) &= 2 \sum_{r=1}^{kn} r - \sum_{r=1}^{kn} 1 \\
 &= kn(kn+1) - kn = k^2n^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \sum_{r=1}^{5n} (2r-1) &= 25n^2 \Rightarrow \sum_{r=1}^n r^3 = 25n^2 \\
 &\Rightarrow \frac{1}{4}n^2(n+1)^2 = 25n^2 \\
 &\Rightarrow (n+1)^2 = 100 \\
 &\Rightarrow n = 9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{14\ a} \quad \sum_{r=1}^n (r^3 - r^2) &= \sum_{r=1}^n r^3 - \sum_{r=1}^n r^2 \\
 &= \frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1) \\
 &= \frac{1}{12}n(n+1)(3n(n+1) - 2(2n+1)) \\
 &= \frac{1}{12}n(n+1)(3n^2 - n - 2) \\
 &= \frac{1}{12}n(n+1)(n-1)(3n+2)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{14\ b} \quad \sum_{r=1}^n 7r &= \frac{7}{2}n(n+1) \\
 \Rightarrow \frac{7}{2}n(n+1) &= \frac{1}{12}n(n+1)(n-1)(3n+2) \\
 \Rightarrow \frac{7}{2} &= \frac{1}{12}(n-1)(3n+2) \\
 \Rightarrow 42 &= (n-1)(3n+2) \\
 \Rightarrow 3n^2 - n - 44 &= 0 \\
 \Rightarrow (3n+11)(n-4) &= 0 \\
 \Rightarrow n &= 4
 \end{aligned}$$

Challenge

$$\mathbf{a} \quad \text{Let } f_2(x) = ax + b$$

$$\sum_{r=1}^n f_2(r) = \frac{1}{2}an(n+1) + bn$$

$$\Rightarrow \frac{1}{2}an(n+1) + bn = n^2$$

$$\Rightarrow \frac{1}{2}an^2 + \left(\frac{1}{2}a + b\right)n$$

$$a = 2, b = -1$$

$$\Rightarrow f_2(x) = 2x - 1$$

$$\text{Let } f_3(x) = ax^2 + bx + c$$

$$\sum_{r=1}^n f_3(r) = \frac{1}{6}an(n+1)(2n+1) + \frac{1}{2}bn(n+1) + cn$$

$$\Rightarrow \frac{1}{6}an(n+1)(2n+1) + \frac{1}{2}bn(n+1) + cn = n^3$$

$$\Rightarrow \frac{1}{3}an^3 + \left(\frac{1}{2}a + \frac{1}{2}b\right)n^2 + \left(\frac{1}{6}a + \frac{1}{2}b + c\right)n = n^3$$

$$a = 3, b = -3, c = 1$$

$$\Rightarrow f_3(x) = 3x^2 - 3x + 1$$

$$\text{Let } f_4(x) = ax^3 + bx^2 + cx + d$$

$$\sum_{r=1}^n f_4(r) = \frac{1}{4}an^2(n+1)^2 + \frac{1}{6}bn(n+1)(2n+1)$$

$$+ \frac{1}{2}cn(n+1) + dn$$

$$\Rightarrow \frac{1}{4}a = 1$$

$$\frac{1}{2}a + \frac{1}{3}b = 0$$

$$\frac{1}{4}a + \frac{1}{2}b + \frac{1}{2}c = 0$$

$$\frac{1}{6}b + \frac{1}{2}c + d = 0$$

$$a = 4, b = -6, c = 4, d = -1$$

$$\Rightarrow f_4(x) = 4x^3 - 6x^2 + 4x - 1$$

$$\mathbf{b} \quad \text{Given } h(x) = ax^3 + bx^2 + cx + d,$$

$$nh(n) = an^4 + bn^3 + cn^2 + dn$$

$$= \sum_{r=1}^n (af_4(r) + bf_3(r) + cf_2(r) + df_1(r))$$

$$= \sum_{r=1}^n g(r)$$

$$\text{for } g(r) = af_4(r) + bf_3(r) + cf_2(r) + df_1(r)$$