

## Series 3A

$$1 \text{ a } \sum_{r=0}^3 (2r+1) = 1+3+5+7 \\ = 16$$

$$b \sum_{r=1}^{40} r = \frac{1}{2} \times 40 \times 41 \\ = 820$$

$$c \sum_{r=1}^{20} r = \frac{1}{2} \times 20 \times 21 \\ = 210$$

$$d \sum_{r=1}^{99} r = \frac{1}{2} \times 99 \times 100 \\ = 4950$$

$$e \sum_{r=10}^{40} r = \sum_{r=1}^{40} r - \sum_{r=1}^9 r \\ = \frac{1}{2} \times 40 \times 41 - \frac{1}{2} \times 9 \times 10 \\ = 820 - 45 = 775$$

$$f \sum_{r=100}^{200} r = \sum_{r=1}^{200} r - \sum_{r=1}^{99} r \\ = \frac{1}{2} \times 200 \times 201 - \frac{1}{2} \times 99 \times 100 \\ = 20\,100 - 4950 = 15\,150$$

$$g \sum_{r=21}^{40} r = \sum_{r=1}^{40} r - \sum_{r=1}^{20} r \\ = \frac{1}{2} \times 40 \times 41 - \frac{1}{2} \times 20 \times 21 \\ = 820 - 210 = 610$$

$$h \sum_{r=1}^k r + \sum_{r=k+1}^{80} r = \sum_{r=1}^k r + \sum_{r=1}^{80} r - \sum_{r=1}^k r \\ = \sum_{r=1}^{80} r \\ = \frac{1}{2} \times 80 \times 81 \\ = 3240$$

$$2 \quad \frac{1}{2} \times n \times (n+1) = 528 \\ n^2 + n - 1056 = 0 \\ (n+33)(n-32) = 0 \\ n = 32$$

$$3 \quad \frac{1}{2} k(k+1) = \frac{1}{2} \times \frac{1}{2} \times 20 \times 21 \\ k^2 + k - 210 = 0 \\ (k+15)(k-14) = 0 \\ k = 14$$

$$4 \text{ a } \sum_{r=1}^{2n-1} r = \frac{1}{2} \times (2n-1) \times (2n-1+1) \\ = \frac{1}{2} \times (2n-1) \times 2n \\ = n(2n-1)$$

$$b \sum_{r=n+1}^{2n-1} r = \sum_{r=1}^{2n-1} r - \sum_{r=1}^n r = n(2n-1) - \frac{1}{2}n(n+1) \\ = \frac{1}{2}n(4n-2-n-1) \\ = \frac{1}{2}n(3n-3) = \frac{3}{2}n(n-1)$$

$$5 \quad \sum_{r=n-1}^{2n} r = \sum_{r=1}^{2n} r - \sum_{r=1}^{n-2} r \\ = \frac{1}{2}(2n)(2n+1) - \frac{1}{2}(n-2)(n-1) \\ = \frac{1}{2}(2n(2n+1) - (n-2)(n-1)) \\ = \frac{1}{2}(3n^2 + 5n - 2) \\ = \frac{1}{2}(n+2)(3n-1)$$

$$6 \text{ a } \sum_{r=1}^{n^2} r - \sum_{r=1}^n r \\ = \frac{1}{2}n^2(n^2+1) - \frac{1}{2}n(n+1) \\ = \frac{1}{2}n(n(n^2+1) - (n+1)) \\ = \frac{1}{2}n(n^3 + n - n - 1) \\ = \frac{1}{2}n(n^3 - 1)$$

$$b \sum_{r=10}^{81} r = \sum_{r=1}^{9^2} r - \sum_{r=1}^9 r \\ = \frac{1}{2}n(n^3 - 1) \\ = \frac{1}{2} \times 9 \times (9^3 - 1) \\ = 3276$$

$$\begin{aligned}
 7 \text{ a } \sum_{r=1}^{55} (3r-1) &= 3 \sum_{r=1}^{55} r - \sum_{r=1}^{55} 1 \\
 &= 3 \times \frac{1}{2} \times 55 \times 56 - 55 \\
 &= 4565
 \end{aligned}$$

$$\begin{aligned}
 7 \text{ b } \sum_{r=1}^{90} (2-7r) &= 2 \sum_{r=1}^{90} 1 - 7 \sum_{r=1}^{90} r \\
 &= 2 \times 90 - 7 \times \frac{1}{2} \times 90 \times 91 \\
 &= -28485
 \end{aligned}$$

$$\begin{aligned}
 7 \text{ c } \sum_{r=1}^{46} (9+2r) &= 9 \sum_{r=1}^{46} 1 + 2 \sum_{r=1}^{46} r \\
 &= 9 \times 46 + 2 \times \frac{1}{2} \times 46 \times 47 \\
 &= 2576
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ a } \sum_{r=1}^n (3r+2) &= 3 \sum_{r=1}^n r + 2 \sum_{r=1}^n 1 \\
 &= \frac{3}{2} n(n+1) + 2n \\
 &= \frac{1}{2} n(3(n+1) + 4) = \frac{1}{2} n(3n+7)
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ b } \sum_{r=1}^{2n} (5r-4) &= 5 \sum_{r=1}^{2n} r - 4 \sum_{r=1}^{2n} 1 \\
 &= \frac{5}{2} (2n)(2n+1) - 4(2n) \\
 &= 5n(2n+1) - 8n \\
 &= 10n^2 - 3n = n(10n-3)
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ c } \sum_{r=1}^{n+2} (2r+3) &= 2 \sum_{r=1}^{n+2} r + 3 \sum_{r=1}^{n+2} 1 \\
 &= (n+2)(n+3) + 3(n+2) \\
 &= (n+2)((n+3)+3) \\
 &= (n+2)(n+6)
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ d } \sum_{r=3}^n (4r+5) &= 4 \sum_{r=3}^n 4r + 5 \sum_{r=3}^n 1 \\
 &= 4 \left( \sum_{r=1}^n r - 5 \sum_{r=1}^2 r \right) + 5 \left( \sum_{r=1}^n 1 - \sum_{r=1}^2 1 \right) \\
 &= 4 \left( \frac{1}{2} n(n+1) - 3 \right) + 5(n-2) \\
 &= 2n^2 + 7n - 22 = (2n+11)(n-2)
 \end{aligned}$$

$$\begin{aligned}
 9 \text{ a } \sum_{r=1}^k (4r-5) &= 4 \sum_{r=1}^k r - 5 \sum_{r=1}^k 1 \\
 &= \frac{4}{2} k(k+1) - 5k \\
 &= 2k^2 - 2k - 5k \\
 &= 2k^2 - 3k
 \end{aligned}$$

$$\begin{aligned}
 9 \text{ b } 2k^2 - 3k > 4850 &\Rightarrow 2k^2 - 3k - 4850 > 0 \\
 &\Rightarrow (2k+97)(k-50) > 0 \\
 &\text{so } k > 50 \text{ since } k \text{ is positive} \\
 &\Rightarrow k = 51
 \end{aligned}$$

$$\begin{aligned}
 10 \sum_{r=1}^n ar + b &= a \sum_{r=1}^n r + \sum_{r=1}^n b \\
 &= a \times \frac{1}{2} n(n+1) + bn \\
 &= \frac{1}{2} an^2 + \left( \frac{1}{2} a + b \right) n \\
 &= \frac{1}{2} n(an + (a+2b))
 \end{aligned}$$

$$\begin{aligned}
 \text{Since } \sum_{r=1}^n f(r) &= \frac{1}{2} n(7n+1) \\
 \Rightarrow a &= 7 \\
 \Rightarrow a+2b &= 1 \\
 \Rightarrow 7+2b &= 1 \Rightarrow b = -3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{11\ a} \quad \sum_{r=1}^{4n-1} (3r+1) &= 3 \sum_{r=1}^{4n-1} r + \sum_{r=1}^{4n-1} 1 \\
 &= \frac{3}{2} (4n-1)(4n) + (4n-1) \\
 &= 6n(4n-1) + (4n-1) \\
 &= 24n^2 - 6n + 4n - 1 \\
 &= 24n^2 - 2n - 1
 \end{aligned}$$

**b** Substituting  $n = 25$  into the result from part **a**:

$$24 \times 25^2 - 2 \times 25 - 1 = 14\,949$$

$$\begin{aligned}
 \mathbf{12\ a} \quad \sum_{r=1}^{2k+1} (4-5r) &= 4 \sum_{r=1}^{2k+1} 1 - 5 \sum_{r=1}^{2k+1} r \\
 &= 4(2k+1) - \frac{5}{2} (2k+1)(2k+2) \\
 &= (2k+1)(4-5(k+1)) \\
 &= (2k+1)(-1-5k) \\
 &= -(2k+1)(5k+1)
 \end{aligned}$$

**b** Substituting  $k = 12$  into the result from part **a**:

$$-25 \times 61 = -1525$$

$$\mathbf{c} \quad \sum_{r=1}^{15} (5r-4) = -\sum_{r=1}^{15} (4-5r)$$

Substituting  $k = 7$  into the result from part **a**:

$$-(-15 \times 36) = 540$$

**13** Let  $f(r) = ar + b$

$$\Rightarrow a \times \frac{1}{2} n(n+1) + bn$$

$$= \frac{1}{2} an^2 + \left(\frac{1}{2}a + b\right)n$$

$$\text{Since } \sum_{r=1}^n f(r) = n^2 + 4n$$

$$\Rightarrow \frac{1}{2}a = 1 \Rightarrow a = 2$$

$$\text{and } \frac{1}{2}a + b = 4 \Rightarrow 1 + b = 4 \Rightarrow b = 3$$

$$\Rightarrow f(r) = 2r + 3$$

$$\begin{aligned}
 \mathbf{14\ a} \quad \sum_{r=1}^n ar + b &= \frac{1}{2} an^2 + \left(\frac{1}{2}a + b\right)n \\
 \sum_{r=1}^4 f(r) &= 36 \Rightarrow 8a + 2a + 4b = 36
 \end{aligned}$$

$$\Rightarrow 10a + 4b = 36 \Rightarrow 5a + 2b = 18$$

$$\sum_{r=1}^6 f(r) = 78 \Rightarrow 18a + 3a + 6b = 78$$

$$\Rightarrow 21a + 6b = 78 \Rightarrow 7a + 2b = 26$$

Solving simultaneously:

$$2a = 8 \Rightarrow a = 4$$

$$5 \times 4 + 2b = 18 \Rightarrow b = -1$$

$$\Rightarrow f(r) = 4r - 1$$

$$\begin{aligned}
 \mathbf{b} \quad \sum_{r=1}^{10} f(r) &= \sum_{r=1}^{10} (2r-1) \\
 &= \frac{1}{2} \times 4 \times 10^2 + \left(\frac{1}{2} \times 4 - 1\right) \times 10 \\
 &= 200 + 10 \\
 &= 210
 \end{aligned}$$

### Challenge

$$\sum_{r=n}^{2n} (12 - 2r) = 0$$

$$\Rightarrow \sum_{r=n}^{2n} 12 - \sum_{r=n}^{2n} 2r = 0$$

$$\sum_{r=n}^{2n} 12 = \sum_{r=n}^{2n} 2r$$

$$\sum_{r=n}^{2n} 12 = \sum_{r=1}^{2n} 12 - \sum_{r=1}^{n-1} 12 = 12(n+1)$$

$$\sum_{r=n}^{2n} 2r = \sum_{r=1}^{2n} 2r - \sum_{r=1}^{n-1} 2r$$

$$= 2 \times \frac{1}{2} \times 2n(2n+1) - 2 \times \frac{1}{2} (n-1)n$$

$$= 3n^2 + 3n$$

$$\Rightarrow 3n^2 + 3n = 12n + 12$$

$$\Rightarrow n^2 - 3n - 4 = 0$$

$$\Rightarrow (n-4)(n+1) = 0$$

$$\Rightarrow n = 4$$