Complex numbers Mixed exercise 1

1 a $z_1 + z_2 = (8-3i) + (-2+4i)$
$= 6 + i$

b $3z_2 = 3(-2+4i)$
$= -6 + 12i$

c $6z_1 - z_2 = 6(8-3i) - (-2+4i)$
$= 48 - 18i + 2 - 4i$
$= 50 - 22i$

2 If $z^2 + bz + 14 = 0$ has no real roots then the discriminant < 0

$$\begin{aligned} b^2 - 4(1)(14) &< 0 \\ b^2 &< 56 \\ -\sqrt{56} < b &< \sqrt{56} \\ -2\sqrt{14} < b &< 2\sqrt{14} \end{aligned}$$

3 $z^2 - 6z + 12 = 0$
Solve by completing the square:

$$\begin{aligned} (z-3)^2 - 9 + 12 &= 0 \\ (z-3)^2 &= -3 \\ z - 3 &= \pm i\sqrt{3} \\ z &= 3 \pm i\sqrt{3} \end{aligned}$$

So $z_1 = 3 + i\sqrt{3}$ and $z_2 = 3 - i\sqrt{3}$

4 To expand $(1+2i)^5$ use the binomial expansion of $(a+b)^5$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

So

$$\begin{aligned} (1+2i)^5 &= (1)^5 + 5(1)^4(2i) + 10(1)^3(2i)^2 \\ &\quad + 10(1)^2(2i)^3 + 5(1)(2i)^4 + (2i)^5 \\ &= 1 + 10i + 40i^2 + 80i^3 + 80i^4 + 32i^5 \end{aligned}$$

Use the fact that $i^2 = -1, i^3 = -i, i^4 = 1$ and $i^5 = i$ to simplify

$$\begin{aligned} (1+2i)^5 &= 1 + 10i - 40 - 80i + 80 + 32i \\ &= 41 - 38i \end{aligned}$$

5 $f(z) = z^2 - 6z + 10$

$$\begin{aligned} f(3+i) &= (3+i)^2 - 6(3+i) + 10 \\ &= 9 + 3i + 3i + i^2 - 18 - 6i + 10 \\ &= (9 - 1 - 18 + 10) + (3 + 3 - 6)i \\ &= 0 \end{aligned}$$

6 a If $z_1 = 4 + 2i$ then z_1^* is the complex conjugate. So $z_1^* = 4 - 2i$

b $z_1 z_2 = (4+2i)(-3+i)$
$= -12 + 4i - 6i + 2i^2$
$= -14 - 2i$

c

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{4+2i}{-3+i} \\ &= \frac{(4+2i)}{(-3+i)} \times \frac{(-3-i)}{(-3-i)} \\ &= \frac{-12 - 4i - 6i - 2i^2}{9 + 3i - 3i - i^2} \\ &= \frac{-10 - 10i}{10} \\ &= -1 - i \end{aligned}$$

7

$$\begin{aligned} (7-2i)^2 &= 49 - 14i - 14i + 4i^2 \\ &= 45 - 28i \end{aligned}$$

$$\begin{aligned} \frac{(7-2i)^2}{1+i\sqrt{3}} &= \frac{(45-28i)}{(1+i\sqrt{3})} \times \frac{(1-i\sqrt{3})}{(1-i\sqrt{3})} \\ &= \frac{45 - 45i\sqrt{3} - 28i + 28i^2\sqrt{3}}{1 - i\sqrt{3} + i\sqrt{3} - (\sqrt{3})^2 i^2} \\ &= \frac{45 - 28\sqrt{3}}{4} + \left(\frac{-28 - 45\sqrt{3}}{4}\right)i \end{aligned}$$

8 $\frac{4-7\text{i}}{z}=3+\text{i}$

$$z=\frac{4-7\text{i}}{3+\text{i}}$$
$$z=\frac{(4-7\text{i})}{3+\text{i}}\times\frac{(3-\text{i})}{(3-\text{i})}$$
$$=\frac{12-4\text{i}-21\text{i}+7\text{i}^2}{9-3\text{i}+3\text{i}-\text{i}^2}$$
$$=\frac{5-25\text{i}}{10}$$
$$=\frac{1}{2}-\frac{5}{2}\text{i}$$

9 $z=\frac{1}{2+\text{i}}$

$$=\frac{1}{(2+\text{i})}\times\frac{(2-\text{i})}{(2-\text{i})}$$
$$=\frac{2-\text{i}}{4-2\text{i}+2\text{i}-\text{i}^2}$$
$$z=\frac{2-\text{i}}{5}=\frac{2}{5}-\frac{1}{5}\text{i}$$

a $z^2=\left(\frac{2-\text{i}}{5}\right)\left(\frac{2-\text{i}}{5}\right)$

$$=\frac{4-2\text{i}-2\text{i}+\text{i}^2}{25}$$

So $z^2=\frac{3}{25}-\frac{4}{25}\text{i}$

b $z-\frac{1}{z}=\left(\frac{2}{5}-\frac{1}{5}\text{i}\right)-\frac{1}{\left(\frac{1}{2+\text{i}}\right)}$

$$=\left(\frac{2}{5}-\frac{1}{5}\text{i}\right)-(2+\text{i})$$
$$=-\frac{8}{5}-\frac{6}{5}\text{i}$$

10 If $z=a+b\text{i}$, then $z^*=a-b\text{i}$

So $\frac{z}{z^*}=\frac{a+b\text{i}}{a-b\text{i}}$

$$=\frac{(a+b\text{i})}{(a-b\text{i})}\times\frac{(a+b\text{i})}{(a+b\text{i})}$$
$$=\frac{a^2+ab\text{i}+ab\text{i}+b^2\text{i}^2}{a^2+ab\text{i}-ab\text{i}-b^2\text{i}^2}$$
$$=\frac{a^2-b^2+2ab\text{i}}{a^2+b^2}$$
$$=\left(\frac{a^2-b^2}{a^2+b^2}\right)+\left(\frac{2ab}{a^2+b^2}\right)\text{i}$$

11 a $z=\frac{3+q\text{i}}{q-5\text{i}}$

Multiply by the complex conjugate

$$z=\frac{(3+q\text{i})}{(q-5\text{i})}\times\frac{(q+5\text{i})}{(q+5\text{i})}$$
$$=\frac{3q+15\text{i}+q^2\text{i}+5q\text{i}^2}{q^2+5q\text{i}-5q\text{i}-25\text{i}^2}$$
$$=\left(\frac{-2q}{q^2+25}\right)+\left(\frac{q^2+15}{q^2+25}\right)\text{i}$$

The real part of z is $\frac{1}{13}$, so

$$\frac{-2q}{q^2+25}=\frac{1}{13}$$
$$-26q=q^2+25$$
$$q^2+26q+25=0$$
$$(q+25)(q+1)=0$$

So $q=-1$ or $q=-25$

b If $q=-1$

$$z=\frac{-2(-1)}{(-1)^2+25}+\frac{(-1)^2+15}{(-1)^2+25}\text{i}$$
$$=\frac{2}{26}+\frac{16}{26}\text{i}$$
$$=\frac{1}{13}+\frac{8}{13}\text{i}$$

If $q=-25$

$$z=\left(\frac{-2(-25)}{(-25)^2+25}\right)+\left(\frac{(-25)^2+15}{(-25)^2+25}\right)\text{i}$$
$$=\frac{50}{650}+\frac{640}{650}\text{i}$$
$$=\frac{1}{13}+\frac{64}{65}\text{i}$$

12 $z+4\mathrm{i}z^{*}=-3+18\mathrm{i}$ **(*)**

Substitute $z=x+\mathrm{i}y$ and $z^{*}=x-\mathrm{i}y$ in **(*)**

$$x+\mathrm{i}y+4\mathrm{i}(x-\mathrm{i}y)=-3+18\mathrm{i}$$
$$x+\mathrm{i}y+4x\mathrm{i}-4y\mathrm{i}^2=-3+18\mathrm{i}$$
$$(x+4y)+(4x+y)\mathrm{i}=-3+18\mathrm{i}$$

Equate real parts and imaginary parts:

$x+4y=-3$ $(\mathbf{1})$

$4x+y=18$ $(\mathbf{2})$

Multiply $(\mathbf{2})$ by 4:

$16x+4y=72$ $(\mathbf{3})$

Subtract $(\mathbf{1})$ from $(\mathbf{3})$

$15x=75$, so $x=5$

Substitute $x=5$ into $(\mathbf{2})$

$4(5)+y=18$, so $y=-2$.

13 $\dfrac{z}{w}=\dfrac{9+6\mathrm{i}}{2-3\mathrm{i}}$

Multiply by the complex conjugate

$$\frac{z}{w}=\frac{(9+6\mathrm{i})}{2-3\mathrm{i}}\times\frac{(2+3\mathrm{i})}{2+3\mathrm{i}}$$
$$=\frac{18+27\mathrm{i}+12\mathrm{i}+18\mathrm{i}^2}{4+6\mathrm{i}-6\mathrm{i}-9\mathrm{i}^2}$$
$$=\frac{39}{13}\mathrm{i}$$
$$=3\mathrm{i}$$

14 $z=\dfrac{q+3\mathrm{i}}{4+q\mathrm{i}}$

Multiply by the complex conjugate

$$z=\frac{(q+3\mathrm{i})}{(4+q\mathrm{i})}\times\frac{(4-q\mathrm{i})}{(4-q\mathrm{i})}$$
$$=\frac{4q-q^2\mathrm{i}+12\mathrm{i}-3q\mathrm{i}^2}{16-4q\mathrm{i}+4q\mathrm{i}-q^2\mathrm{i}^2}$$
$$=\frac{4q+3q}{16+q^2}+\frac{12-q^2}{16+q}\mathrm{i}$$
$$=\frac{7q}{16+q^2}+\frac{12-q^2}{16+q^2}\mathrm{i}$$

15 a If $6-2\mathrm{i}$ is a root, then the complex conjugate $6+2\mathrm{i}$ is also a root.

b $$(z-(6-2\mathrm{i}))(z-(6+2\mathrm{i}))$$
$$=z^2-(6+2\mathrm{i})z-(6-2\mathrm{i})z+(6-2\mathrm{i})(6+2\mathrm{i})$$
$$=z^2-6z-2\mathrm{i}z-6z+2\mathrm{i}z+36+12\mathrm{i}-12\mathrm{i}-4\mathrm{i}^2$$
$$=z^2-12z+36+4$$
$$=z^2-12z+40$$

The equation is $z^2-12z+40=0$

16 If $z=4-k\mathrm{i}$ is a root, then the complex conjugate $4+k\mathrm{i}$ is also a root.

$$(z-(4-k\mathrm{i}))(z-(4+k\mathrm{i}))$$
$$=z^2-(4+k\mathrm{i})z-(4-k\mathrm{i})z+(4-k\mathrm{i})(4+k\mathrm{i})$$
$$=z^2-4z-k\mathrm{i}z-4z+k\mathrm{i}z+16+4k\mathrm{i}-4k\mathrm{i}-k^2\mathrm{i}^2$$
$$=z^2-8z+16+k^2$$

So $z^2-8z+16k^2=z^2-2mz+52$

Equate z coefficients:

$-8=2m$, so $m=4$.

Equate constants:

$16+k^2=52$, so $k^2=36\Rightarrow k=\pm6$

But $k>0$, so $k=6$.

17 If $2+\mathrm{i}$ is a root of $\mathrm{h}(z)=0$ then the complex conjugate $2-\mathrm{i}$ is also a root.

$$\begin{aligned}\mathrm{h}(z)&=(z-(2+\mathrm{i}))(z-(2-\mathrm{i}))\\&=z^2-(2-\mathrm{i})z-(2+\mathrm{i})z+(2+\mathrm{i})(2-\mathrm{i})\\&=z^2-2z+\mathrm{i}z-2z-\mathrm{i}z+4-2\mathrm{i}+2\mathrm{i}-\mathrm{i}^2\\&=z^2-4z+5\end{aligned}$$

Divide to find the remaining factor

$$\begin{array}{r}z+4\\ z^2-4z+5\,\big)\,\overline{z^3+0z^2-11z+20}\\ z^3-4z^2+\ 5z\\ 4z^2-16z+20\\ 4z^2-16z+20\\ 0\end{array}$$

So $\mathrm{h}(z)=(z^2-4z+5)(z+4)=0$

has roots $2+\mathrm{i}$, $2-\mathrm{i}$ and -4

18 If $\mathrm{f}(1+3\mathrm{i})=0$, then $\mathrm{f}(1-3\mathrm{i})=0$.

So $\mathrm{f}(z)=(z-(1+3\mathrm{i}))(z-(1-3\mathrm{i}))$

$$\begin{aligned}&=z^2-(1-3\mathrm{i})z-(1+3\mathrm{i})z+(1+3\mathrm{i})(1-3\mathrm{i})\\&=z^2-z+3\mathrm{i}z-z-3\mathrm{i}z+1-3\mathrm{i}+3\mathrm{i}-9\mathrm{i}^2\\&=z^2-2z+10\end{aligned}$$

Divide to find the remaining factor

$$\begin{array}{r}z+2\\ z^2-2z+10\,\big)\,\overline{z^3+0z^2+\ 6z+20}\\ z^3-2z^2+10z\\ 2z^2-\ 4z+20\\ 2z^2-\ 4z+20\\ 0\end{array}$$

So $z^3+62+20=(z^2-2z+10)(z+2)$

either $(z^2-2z+10)=0\Rightarrow z=1\pm3\mathrm{i}$

(as these were the roots given at the beginning of the question)

or $z+2=0\Rightarrow z=-2$

So the roots of $z^3+6z+20=0$ are $1+3\mathrm{i}$, $1-3\mathrm{i}$ and -2

19 a $\mathrm{f}(z)=z^3+3z^2+kz+48$

If $\mathrm{f}(4\mathrm{i})=0$, then:

$$\begin{aligned}0&=\mathrm{f}(4\mathrm{i})\\&=(4\mathrm{i})^3+3(4\mathrm{i})^2+k(4\mathrm{i})+48\\&=64\mathrm{i}^3+48\mathrm{i}^2+4k\mathrm{i}+48\\&=-64\mathrm{i}-48+4k\mathrm{i}+48\\&=(4k-64)\mathrm{i}\end{aligned}$$

Hence $4k-64=0\Rightarrow k=16$.

b If $4\mathrm{i}$ is a root then $-4\mathrm{i}$ is also a root and $(z-4\mathrm{i})(z-(-4\mathrm{i}))$ is a quadratic factor of $\mathrm{f}(z)$

Now $(z-4\mathrm{i})(z-(-4\mathrm{i}))=(z-4\mathrm{i})(z+4\mathrm{i})$

$$\begin{aligned}&=z^2+4\mathrm{i}z-4\mathrm{i}z-16\mathrm{i}^2\\&=z^2+16\end{aligned}$$

Divide to find the remaining factor

$$\begin{array}{r}z+3\\ z^2+0z+16\,\big)\,\overline{z^3+3z^3+16z+48}\\ z^3+0z^2+16z\\ 3z^2+\ 0z+48\\ 3z^2+\ 0z+48\\ 0\end{array}$$

So $\mathrm{f}(z)=(z^2+16)(z+3)=0$ has two other roots which are $-4\mathrm{i}$ and -3

20 a $z^4-z^3-16z^2-74z-60$
$$=(z^2-5z-6)(z^2+bz+c)$$

Equate z^3 coefficients:

$b-5=-1$, so $b=4$.

Equate constants:

$-6c=-60$, so $c=10$.

20 b $0 = \mathrm{f}(z)$

$= (z^2 - 5z - 6)(z^2 + 4z + 10)$

$= (z - 6)(z + 1)(z^2 + 4z + 10)$

Either $z - 6 = 0 \Rightarrow z = 6$

or $z + 1 = 0 \Rightarrow z = -1$

or $z^2 + 4z + 10 = 0$

$(z + 2)^2 - 4 + 10 = 0$

$(z + 2)^2 = -6$

$z + 2 = \pm\mathrm{i}\sqrt{6}$

$z = -2 \pm \mathrm{i}\sqrt{6}$

So the roots of $\mathrm{f}(z) = 0$ are $6,\ -1,\ -2 + \mathrm{i}\sqrt{6}$ and $-2 - \mathrm{i}\sqrt{6}.$

21 If $\mathrm{g}(3 - 2\mathrm{i}) = 0$ then $\mathrm{g}(3 + 2\mathrm{i}) = 0$ and $(z - (3 - 2\mathrm{i}))(z - (3 + 2\mathrm{i}))$ is a quadratic factor of $\mathrm{g}(z)$

Now $(z - (3 - 2\mathrm{i}))(z - (3 + 2\mathrm{i}))$

$= z^2 - (3 + 2\mathrm{i})z - (3 - 2\mathrm{i})z + (3 - 2\mathrm{i})(3 + 2\mathrm{i})$

$= z^2 - 3z - 2\mathrm{i}z - 3z + 2\mathrm{i}z + 9 + 6\mathrm{i} - 6\mathrm{i} - 4\mathrm{i}^2$

$= z^2 - 6z + 9 + 4$

$= z^2 - 6z + 13$

Divide to find the remaining factor

$$
\begin{array}{r|l}
 & z^2 + 6 \\
\hline
z^2 - 6z + 13 & z^4 - 6z^3 + 19z^2 - 36z + 78 \\
 & z^4 - 6z^3 + 13z^2 \\
 & \qquad\quad 6z^2 - 36z + 78 \\
 & \qquad\quad 6z^2 - 36z + 78 \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

$0 = \mathrm{g}(z)$

$= (z^2 + 6)(z^2 - 6z + 13)$

Either $z^2 - 6z + 13 = 0$

$\Rightarrow z = 3 \pm 2\mathrm{i}$

(as these were the two original roots)

or $z^2 + 6 = 0$

$z^2 = -6$

$z = \pm\mathrm{i}\sqrt{6}$

21 (cont.)

So the roots of $\mathrm{g}(z) = 0$ are

$3 + 2\mathrm{i},\ 3 - 2\mathrm{i},\ \mathrm{i}\sqrt{6}$ and $-\mathrm{i}\sqrt{6}.$

22 a $\mathrm{f}(z) = z^4 - 2z^3 - 5z^2 + pz + 24$

$0 = \mathrm{f}(4)$

$= 4^4 - 2(4)^3 - 5(4)^2 + 4p + 24$

$= 256 - 128 - 80 + 4p + 24$

$4p = -72$

$p = -18$

b If $\mathrm{f}(4) = 0$ then $(z - 4)$ is a factor of $\mathrm{f}(z)$

Divide to find the remaining factors

$$
\begin{array}{r|l}
 & z^3 + 2z^3 + 3z - 6 \\
\hline
z - 4 & z^4 - 2z^3 - 5z^2 - 18z + 24 \\
 & z^4 - 4z^3 \\
 & \qquad 2z^3 - 5z^2 \\
 & \qquad 2z^3 - 8z^2 \\
 & \qquad\qquad 3z^2 - 18z \\
 & \qquad\qquad 3z^3 - 12z \\
 & \qquad\qquad\qquad -6z + 24 \\
 & \qquad\qquad\qquad -6z + 24 \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

So $0 = \mathrm{f}(z)$

$= (z - 4)(z^3 + 2z^2 + 3z - 6)$

Either $z - 4 = 0 \Rightarrow z = 4$

or $z^3 + 2z^2 + 3z - 6 = 0$ **(*)**

Now **(*)** is a cubic equation, so must have at least one real root (since complex roots always come in pairs).
Try small real values of z to find a factor of **(*)**:

Let $z = 1$: $(1)^3 + 2(1)^2 + 3(1) - 6 = 0$

So $z = 1$ is a root of **(*)**, and hence $z - 1$ is a factor.

So $z^3 + 2z^2 + 3z - 6 = (z - 1)(z^2 + bz + c)$

Equate z^2 terms:

$-1 + b = 2$, so $b = 3$

Equate constants:

$-6 = -c$, so $c = 6$

22 b (cont.)

Hence

$z^3+2z^2+3z-6=(z-1)(z^2+3z+6)$

Either $z=1$

or

$$(z^2+3z+6)=0$$
$$\left(z+\frac{3}{2}\right)^2-\frac{9}{4}+6=0$$
$$\left(z+\frac{3}{2}\right)^2=-\frac{15}{4}$$
$$z+\frac{3}{2}=\pm\frac{\sqrt{15}}{2}\mathrm{i}$$
$$z=-\frac{3}{2}\pm\frac{\sqrt{15}}{2}\mathrm{i}$$

So the roots of $\mathrm{f}(z)=0$ are

$1,\ 4,\ \left(-\frac{3}{2}+\frac{\sqrt{15}}{2}\mathrm{i}\right)$ and $\left(-\frac{3}{2}-\frac{\sqrt{15}}{2}\mathrm{i}\right)$

Challenge

a Suppose, for a contradiction, that a cubic equation has repeated non-real roots. This repeated root gives us two roots of the cubic.
The complex conjugate of each repeated root would also be a root of the cubic.
This means we have found four roots of the cubic.
However, this is a contradiction, as a cubic has exactly three roots in total.
Therefore, a cubic equation cannot have repeated non-real roots.

b Any answer of the form $(z+a\mathrm{i})^2(z-a\mathrm{i})^2$ will work.

$(z+a\mathrm{i})^2(z-a\mathrm{i})^2$

$=(z^2+2za\mathrm{i}+a^2\mathrm{i}^2)(z^2-2za\mathrm{i}+a^2\mathrm{i}^2)$

$=z^4-2a\mathrm{i}z^3+a^2\mathrm{i}^2z^2+2a\mathrm{i}z^3-4a^2\mathrm{i}^2z^2$
$\quad+2a^3\mathrm{i}^3z+a^2\mathrm{i}^2z^2-2a^3\mathrm{i}^3z+a^4\mathrm{i}^4$

$=z^4-2az^3\mathrm{i}-a^2z^2+2az^3\mathrm{i}+4a^2z^2$
$\quad-2a^3z\mathrm{i}-a^2z^2+2a^3z\mathrm{i}+a^4$

$=z^4+2a^2z^2+a^4$

This is a quadratic equation with real coefficients. Its roots are non-real and repeated.