Complex numbers 1C

1 a

$$\begin{aligned}(5+\mathrm{i})(3+4\mathrm{i}) &= 5(3+4\mathrm{i})+\mathrm{i}(3+4\mathrm{i})\\ &= 15+20\mathrm{i}+3\mathrm{i}+4\mathrm{i}^2\\ &= 15+20\mathrm{i}+3\mathrm{i}-4\\ &= (15-4)+\mathrm{i}(20+3)\\ &= 11+23\mathrm{i}\end{aligned}$$

b

$$\begin{aligned}(6+3\mathrm{i})(7+2\mathrm{i}) &= 6(7+2\mathrm{i})+3\mathrm{i}(7+2\mathrm{i})\\ &= 42+12\mathrm{i}+21\mathrm{i}+6\mathrm{i}^2\\ &= 42+12\mathrm{i}+21\mathrm{i}-6\\ &= (42-6)+\mathrm{i}(12+21)\\ &= 36+33\mathrm{i}\end{aligned}$$

c

$$\begin{aligned}(5-2\mathrm{i})(1+5\mathrm{i}) &= 5(1+5\mathrm{i})-2\mathrm{i}(1+5\mathrm{i})\\ &= 5+25\mathrm{i}-2\mathrm{i}-10\mathrm{i}^2\\ &= 5+25\mathrm{i}-2\mathrm{i}+10\\ &= (5+10)+\mathrm{i}(25-2)\\ &= 15+23\mathrm{i}\end{aligned}$$

d

$$\begin{aligned}(13-3\mathrm{i})(2-8\mathrm{i}) &= 13(2-8\mathrm{i})-3\mathrm{i}(2-8\mathrm{i})\\ &= 26-104\mathrm{i}-6\mathrm{i}+24\mathrm{i}^2\\ &= 26-104\mathrm{i}-6\mathrm{i}-24\\ &= (26+24)+\mathrm{i}(104-6)\\ &= 2-110\mathrm{i}\end{aligned}$$

e

$$\begin{aligned}(-3-\mathrm{i})(4+7\mathrm{i}) &= -3(4+7\mathrm{i})-\mathrm{i}(4+7\mathrm{i})\\ &= -12-21\mathrm{i}-4\mathrm{i}-7\mathrm{i}^2\\ &= -12-21\mathrm{i}-4\mathrm{i}+7\\ &= (-12+7)+\mathrm{i}(-21-4)\\ &= -5-25\mathrm{i}\end{aligned}$$

f

$$\begin{aligned}(8+5\mathrm{i})(8+5\mathrm{i}) &= 8(8+5\mathrm{i})+5\mathrm{i}(8+5\mathrm{i})\\ &= 64+40\mathrm{i}+40\mathrm{i}+25\mathrm{i}^2\\ &= 64+40\mathrm{i}+40\mathrm{i}-25\\ &= (64-25)+\mathrm{i}(40+40)\\ &= 39+80\mathrm{i}\end{aligned}$$

1 g

$$\begin{aligned}(2-9\mathrm{i})(2-9\mathrm{i}) &= 2(2-9\mathrm{i})-9\mathrm{i}(2-9\mathrm{i})\\ &= 4-18\mathrm{i}-18\mathrm{i}+81\mathrm{i}^2\\ &= 4-18\mathrm{i}-18\mathrm{i}-81\\ &= (4-81)+\mathrm{i}(-18-18)\\ &= -77-36\mathrm{i}\end{aligned}$$

h

$$\begin{aligned}&(1+\mathrm{i})(2+\mathrm{i})(3+\mathrm{i})\\ &= (1+\mathrm{i})[2(3+\mathrm{i})+\mathrm{i}(3+\mathrm{i})]\\ &= (1+\mathrm{i})[6+2\mathrm{i}+3\mathrm{i}+\mathrm{i}^2]\\ &= (1+\mathrm{i})[6+2\mathrm{i}+3\mathrm{i}-1]\\ &= (1+\mathrm{i})(5+5\mathrm{i})\\ &= 1(5+5\mathrm{i})+\mathrm{i}(5+5\mathrm{i})\\ &= 5+5\mathrm{i}+5\mathrm{i}+5\mathrm{i}^2\\ &= 5+5\mathrm{i}+5\mathrm{i}-5\\ &= 10\mathrm{i}\end{aligned}$$

i

$$\begin{aligned}&(3-2\mathrm{i})(5+\mathrm{i})(4-2\mathrm{i})\\ &= (3-2\mathrm{i})[5(4-2\mathrm{i})+\mathrm{i}(4-2\mathrm{i})]\\ &= (3-2\mathrm{i})[20-10\mathrm{i}+4\mathrm{i}-2\mathrm{i}^2]\\ &= (3-2\mathrm{i})[20-10\mathrm{i}+4\mathrm{i}+2]\\ &= (3-2\mathrm{i})[22-6\mathrm{i}]\\ &= 3(22-6\mathrm{i})-2\mathrm{i}(22-6\mathrm{i})\\ &= 66-18\mathrm{i}-44\mathrm{i}+12\mathrm{i}^2\\ &= 66-18\mathrm{i}-44\mathrm{i}-12\\ &= 54-62\mathrm{i}\end{aligned}$$

1 j

$$\begin{aligned}(2+3i)^3 &= (2+3\mathrm{i})[(2+3\mathrm{i})(2+3\mathrm{i})]\\ &= (2+3\mathrm{i})[2(2+3\mathrm{i})+3\mathrm{i}(2+3\mathrm{i})]\\ &= (2+3\mathrm{i})[4+6\mathrm{i}+6\mathrm{i}+9\mathrm{i}^2]\\ &= (2+3\mathrm{i})[4+6\mathrm{i}+6\mathrm{i}-9]\\ &= (2+3\mathrm{i})(-5+12\mathrm{i})\\ &= 2(-5+12\mathrm{i})+3\mathrm{i}(-5+12\mathrm{i})\\ &= -10+24\mathrm{i}-15\mathrm{i}+36\mathrm{i}^2\\ &= -10+24\mathrm{i}-15\mathrm{i}-36\\ &= -46+9\mathrm{i}\end{aligned}$$

2 a $(4+5\text{i})(4-5\text{i}) = 16-20\text{i}+20\text{i}-25\text{i}^2$
$= 16-20\text{i}+20\text{i}+25$
$= 41$

b $(7-2\text{i})(7+2\text{i}) = 49+14\text{i}-14\text{i}-4\text{i}^2$
$= 49+14\text{i}-14\text{i}+4$
$= 53$

c The answers to **2a** and **2b** are both real.

d Let a and b be any real numbers.
$(a+b\text{i})(a-b\text{i}) = a^2-ab\text{i}+ab\text{i}-b^2\text{i}^2$
$= a^2-ab\text{i}+ab\text{i}+b^2$
$= a^2+b^2$

For any a and b, the imaginary parts cancel (i.e. they sum to zero), so the answer is always real.

3 $(a+3\text{i})(1+b\text{i}) = 25-39i$
$a+ab\text{i}+3\text{i}+3b\text{i}^2 = 25-39\text{i}$
$a+ab\text{i}+3\text{i}-3b = 25-39\text{i}$
$(a-3b)+(ab+3)\text{i} = 25-39\text{i}$

Equating real parts:
$a-3b = 25$
$a = 3b+25$ **(1)**
Equating imaginary parts:
$ab+3 = -39$ **(2)**

Substituting **(1)** into **(2)**:
$(3b+25)b+3 = -39$
$3b^2+25b+3 = -39$
$3b^2+25b+42 = 0$
$(3b+7)(b+6) = 0$
So $b = \frac{-7}{3}$ or $b = -6$

Substituting $b=-6$ into **(1)**:
$a(-6)+3 = -39$
$a = 7$
Substituting $b = \frac{-7}{3}$ into **(1)**:
$a\left(-\frac{7}{3}\right)+3 = -39$
$a = 18$

3 Hence, the two pairs of values are:
$a = 7,\ b = -6$
$a = 18,\ b = -\frac{7}{3}$

4 a $\text{i}\times\text{i}\times\text{i}\times\text{i}\times\text{i}\times\text{i} = \text{i}^2\times\text{i}^2\times\text{i}^2$
$= -1\times-1\times-1$
$= -1$

b $3\text{i}\times3\text{i}\times3\text{i}\times3\text{i} = 81(\text{i}\times\text{i}\times\text{i}\times\text{i})$
$= 81(\text{i}^2\times\text{i}^2)$
$= 81(-1\times-1)$
$= 81$

c $(\text{i}\times\text{i}\times\text{i}\times\text{i}\times\text{i})+\text{i} = (\text{i}^2\times\text{i}^2\times\text{i})+\text{i}$
$= (-1\times-1\times\text{i})+\text{i}$
$= \text{i}+\text{i}$
$= 2\text{i}$

d $(4\text{i})^3-4\text{i}^3 = (4\text{i}\times4\text{i}\times4\text{i})-4(\text{i}\times\text{i}\times\text{i})$
$= 64(\text{i}\times\text{i}\times\text{i})-4(\text{i}\times\text{i}\times\text{i})$
$= 60(\text{i}\times\text{i}\times\text{i})$
$= 60(-1\times\text{i})$
$= -60\text{i}$

5 To expand $(1+\text{i})^6$, use the binomial expansion of $(a+b)^6$
$(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^2$
$+15a^2b^4+6ab^5+b^6$

Substitute $a=1$ and $b=\text{i}$:
$(1+\text{i})^6 = (1)^6+6(1)^5(\text{i})+15(1)^4(\text{i})^2+20(1)^3(\text{i})^3$
$+15(1)^2(\text{i})^4+6(1)(\text{i})^5+(\text{i})^6$
$(1+\text{i})^6 = 1+6\text{i}+15\text{i}^2+20\text{i}^3+15\text{i}^4+6\text{i}^5+\text{i}^6$

[Use $\text{i}^2=-1, \text{i}^3=-\text{i}, \text{i}^4=1, \text{i}^5=\text{i}$ and $\text{i}^6=-1$]

$(1+\text{i})^6 = 1+6\text{i}-15-20\text{i}+15+6\text{i}-1$
$= -8\text{i}$

So $a=0$ and $b=-8$

6 To expand $(3-2i)^4$, use the binomial expansion of $(a+b)^4$:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Substitute $a=3$ and $b=2i$:

$$(3-2i)^4 = (3)^4 + 4(3)^3(-2i) + 6(3)^2(-2i)^2 + 4(3)(-2i)^3 + (-2i)^4$$

$$(3-2i)^4 = 81 - 216i + 216i^2 - 96i^3 + 16i^4$$

[Use $i^2=-1, i^3=-i$ and $i^4=1$]

$$(3-2i)^4 = 81 - 216i - 216 + 96i + 16$$
$$= -119 - 120i$$

So the real part of $(3-2i)^4$ is -119

7 a $f(2i) = 2(2i)^2 - (2i) + 8$
$= 2(4i^2) - 2i + 8$
$= 8(-1) - 2i + 8$
$= -2i$

b $f(3-6i)$
$= 2(3-6i)^2 - (3-6i) + 8$
$= 2(9 - 36i + 36i^2) - 3 + 6i + 8$
$= 18 - 72i + 72i^2 - 3 + 6i + 8$
$= 18 - 72i - 72 - 3 + 6i + 8$
$= -49 - 66i$

8 $f(1-4i) = (1-4i)^2 - 2(1-4i) + 17$
$= 1 - 8i + 16i^2 - 2 + 8i + 17$
$= 1 - 8i - 16 - 2 + 8i + 17$
$= 0$

$f(1-4i) = 0 \Rightarrow z = 1-4i$ is a solution of $f(z)=0$

9 a $i^3 = (i^2)(i) = (-1)(i) = -i$
$i^4 = (i^2)(i^2) = (-1)(-1) = 1$

9 b $i^5 = (i^4)(i) = (1)(i) = i$
$i^6 = (i^4)(i^2) = (1)(-1) = -1$
$i^7 = (i^4)(i^3) = (1)(-i) = -i$
$i^8 = (i^4)(i^4) = (1)(1) = 1$

c For $n \in \mathbb{N}$,
$i^{4n} = 1,\ i^{4n+1} = i,\ i^{4n+2} = -1,\ i^{4n+3} = -i$

Hence:

i $100 = 4 \times 25 \Rightarrow i^{100} = 1$

ii $253 = (4 \times 63) + 1 \Rightarrow i^{253} = i$

iii $301 = (4 \times 75) + 1 \Rightarrow i^{301} = i$

Challenge

a $(a+bi)^2 = (a+bi)(a+bi)$
$= a^2 + abi + abi + b^2i^2$
$= a^2 + 2abi + b^2(-1)$
$= (a^2 - b^2) + 2abi$

b Let $a^2 - b^2 = 40$ **(1)**
and $2ab = -42$ **(2)**

Then by part **a**, $\sqrt{40-42i} = a + bi$

Rearranging (**2**) gives $a = -\frac{21}{b}$ (**3**)

Substituting (**3**) into (**1**) gives:

$$\left(-\frac{21}{b}\right)^2 - b^2 = 40$$
$$\frac{441}{b^2} - b^2 = 40$$
$$b^4 + 40b^2 - 441 = 0$$
$$(b^2 - 9)(b^2 + 49) = 0$$

b is real, so $b^2 \neq -49$

Hence $b^2 - 9 = 0 \Rightarrow b = -3$

Substituting $b = -3$ into (**3**):

$$a = -\frac{21}{b} = \frac{-21}{-3} = 7$$

So $\sqrt{40-42i} = 7 - 3i$