Further kinematics 8C

1 a $a = 1 - \sin \pi t \text{ ms}^{-2}$, $t = 0$ s, $v = 0 \text{ ms}^{-1}$, $s = 0$ m

$v = \int a \, dt$

$v = \int (1 - \sin \pi t) \, dt$

$v = t + \frac{\cos \pi t}{\pi} + c$

Substituting $v = 0$ when $t = 0$ gives:

$0 = 0 + \frac{\cos 0}{\pi} + c$

$c = -\frac{1}{\pi}$

$\Rightarrow v = t + \frac{\cos \pi t}{\pi} - \frac{1}{\pi}$

b Using expression for v, above:

$s = \int v \, dt$

$s = \int \left(t + \frac{\cos \pi t}{\pi} - \frac{1}{\pi} \right) dt$

$s = \frac{t^2}{2} + \frac{\sin \pi t}{\pi^2} - \frac{t}{\pi} + c$

Substituting $s = 0$ when $t = 0$ gives:

$0 = 0 + 0 - 0 + c$

$c = 0$

$\Rightarrow s = \frac{t^2}{2} + \frac{\sin \pi t}{\pi^2} - \frac{t}{\pi}$

2 a $a = \sin 3\pi t \text{ ms}^{-2}$, $t = 0$ s, $v = \frac{1}{3\pi} \text{ ms}^{-1}$, $s = 1$ m

$v = \int a \, dt$

$v = \int \sin 3\pi t \, dt$

$v = -\frac{\cos 3\pi t}{3\pi} + c$

Using values given:

$\frac{1}{3\pi} = -\frac{\cos 0}{3\pi} + c$

$c = \frac{1}{3\pi} + \frac{1}{3\pi}$

$\Rightarrow v = \frac{2}{3\pi} - \frac{\cos 3\pi t}{3\pi}$

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2 b The maximum value of v occurs when $\cos 3\pi t$ has its minimum value, i.e. −1.

$$v_{\text{max}} = \frac{2}{3\pi} - \frac{-1}{3\pi} = \frac{1}{\pi}$$

The maximum value of v is $\frac{1}{\pi}$ ms^{-1}

c Using expression for v, above:

$$s = \int v \, dt$$

$$s = \int \left(\frac{2}{3\pi} - \frac{\cos 3\pi t}{3\pi} \right) dt$$

$$s = \frac{2t}{3\pi} - \frac{\sin 3\pi t}{9\pi^2} + c$$

Using values given:

$$1 = 0 - 0 + c$$

$$c = 1$$

$$\Rightarrow s = \frac{2t}{3\pi} - \frac{\sin 3\pi t}{9\pi^2} + 1$$

3 a $a = -\cos 4\pi t$ ms^{-2}, $t = 0$ s, $v = 0$ ms^{-1}, $s = 0$ m

$$v = \int a \, dt$$

$$v = \int -\cos 4\pi t \, dt$$

$$v = -\frac{\sin 4\pi t}{4\pi} + c$$

Using values given:

$$0 = -0 + c$$

$$c = 0$$

$$\Rightarrow v = -\frac{\sin 4\pi t}{4\pi}$$

b The maximum value of v occurs when $\sin 4\pi t$ has its minimum value, i.e. −1.

$$v_{\text{max}} = -\frac{-1}{4\pi}$$

The maximum value of v is $\frac{1}{4\pi}$ ms^{-1}

c Using expression for v, above:

$$s = \int v \, dt$$

$$s = \int -\frac{\sin 4\pi t}{4\pi} dt$$

$$s = \frac{\cos 4\pi t}{16\pi^2} + c$$

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3 c Using values given:

$$0 = \frac{1}{16\pi^2} + c$$

$$c = -\frac{1}{16\pi^2}$$

$$\Rightarrow s = \frac{\cos 4\pi t}{16\pi^2} - \frac{1}{16\pi^2}$$

d The maximum value of s occurs when $\cos 4\pi t$ is -1.

$$s_{\max} = -\frac{1}{16\pi^2} - \frac{1}{16\pi^2}$$

The maximum distance from O is $\frac{1}{8\pi^2}$ ms^{-1}

e The particle changes direction when $4\pi t = n\pi$ where n is a whole number.

It therefore changes direction whenever $t = \frac{n\pi}{4\pi}$ s i.e. every 0.25 s.

Between 0 and 4 s it therefore changes direction $\frac{4}{0.25} - 1 = 15$ times (it is stationary at 0 and 4 s).

4 a $v = \frac{ds}{dt}$

$$v = \frac{2}{3}3t^{-\frac{1}{3}} + (-3 \times 2e^{-3t})$$

$$v = 2t^{-\frac{1}{3}} - 6e^{-3t}$$

At $t = 0.5$ s:

$$v = 2\left(0.5^{-\frac{1}{3}}\right) - 6e^{-1.5}$$

$$v = 1.1810...$$

At $t = 0.5$ s, the velocity of M is 1.18 ms^{-1} (3s.f.).

b $a = \frac{dv}{dt}$

$$a = \left(-\frac{1}{3}\right)2t^{-\frac{4}{3}} - (-3 \times 6e^{-3t})$$

$$a = -\frac{2}{3}t^{-\frac{4}{3}} + 18e^{-3t}$$

At $t = 3$ s:

$$a = -\frac{2}{3}\left(3^{-\frac{4}{3}}\right) + 18e^{-9}$$

$$a = -0.15185...$$

At $t = 3$ s, the acceleration of M is -0.152 ms^{-2} (3s.f.).

c Using Newton's second law of motion when $t = 3$ s

$$F = ma$$
$$= 5 \times (-0.152)$$
$$= -0.759 \text{ N (3s.f.)}.$$

So F acts in opposition to the direction of motion.

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5 a At $t = 4$ s, the relevant equation is $s = \frac{t}{2}$. Since $v = \frac{ds}{dt}$

$v = \frac{1}{2}$

At $t = 4$ s, the velocity of P is 0.5 ms^{-1}

b At $t = 22$ s, the relevant equation is $s = \sqrt{t+3}$. Since $v = \frac{ds}{dt}$

$v = \frac{1}{2}(t+3)^{-\frac{1}{2}}$

$v = \frac{1}{2} \times \frac{1}{\sqrt{25}} = \frac{1}{10}$

At $t = 4$ s, the velocity of P is 0.1 ms^{-1}

6 a $t = 2$ s, $s = 3^t + 3t$ m

$v = \frac{ds}{dt}$

$v = 3^t \ln 3 + 3$

$v(2) = 9 \ln 3 + 3$

$v(2) = 12.887...$

At $t = 2$ s, the velocity of P is 12.9 ms^{-1} (3s.f.).

b $t = 10$ s, $s = -252 + 96t - 6t^2$ m

$v = \frac{ds}{dt}$

$v = 96 - 12t$

$v(10) = 96 - 120$

$v(10) = -24$

At $t = 10$ s, the velocity of P is −24 ms^{-1}

c For $0 \le t \le 3$, the displacement is $s = (3^t + 3t)$ m, which is always positive and increasing for $0 \le t \le 3$, so maximum displacement does not occur then.

For $3 < t \le 6$, the displacement is $s = (24t - 36)$ m, which is also always positive and increasing for $3 < t \le 6$, so maximum displacement does not occur then.

Therefore maximum displacement must occur when $t \ge 6$ s.

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6 c For $t > 6$, max displacement occurs when

$$0 = \frac{\mathrm{d}s}{\mathrm{d}t}$$
$$0 = 96 - 12t$$
$$12t = 96$$
$$t = 8$$

Note that $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -12 < 0 \Rightarrow t = 8$ is a max.

$$s(8) = -252 + (96 \times 8) - (6 \times 8^2)$$
$$= -252 + 768 - 384$$
$$= 132$$

The maximum displacement of P is 132 m.

d Check to see if there is a value of t for $0 \le t \le 3$ for which $\frac{\mathrm{d}s}{\mathrm{d}t}$ = 18 ms^{-1}:

$$18 = 3^t \ln 3 + 3$$
$$3^t = \frac{18 - 3}{\ln 3}$$
$$t \ln 3 = \ln 15 - \ln(\ln 3)$$
$$t = \frac{\ln 15 - \ln(\ln 3)}{\ln 3} = 2.3793...$$

At $t = 2.739$ s,

$$s(2.379) = 3^{2.379} + (3 \times 2.379)$$
$$= 20.791...$$

For $3 < t \le 6$, $24t - 36 = \pm 18 \Rightarrow t = 19.5$ s or $t = 0.75$ s, both of which do not lie in the interval $3 < t \le 6$, so no values of t in this interval for qwhich speed is 18 ms^{-1}

For $t > 6$,

$$\pm 18 = 96 - 12t$$
$$12t = 96 \pm 18$$
$$t = 6.5 \text{ and } t = 9.5$$

$$s(6.5) = -252 + (96 \times 6.5) - (6 \times 6.5^2)$$
$$= -252 + 624 - 253.5$$
$$= 118.5$$

$$s(9.5) = -252 + (96 \times 9.5) - (6 \times 9.5^2)$$
$$= -252 + 912 - 541.5$$
$$= 118.5$$

P has a speed of 18 ms^{-1} at displacements of 20.8 m (3s.f.) and 118.5 m (twice).

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7 We will integrate twice to find an expression for the displacement, then find how long it takes to travel 16 m.

$a = 3\sqrt{t}$ ms^{-2}, $t = 1$ s, $v = 2$ ms^{-1}

$$v = \int a\,\mathrm{d}t$$

$$v = \int 3\sqrt{t}\,\mathrm{d}t$$

$$v = 3 \times \frac{2}{3} t^{\frac{3}{2}} + c$$

Using values given:

$$2 = \left(2 \times 1^{\frac{3}{2}}\right) + c$$

$$c = 2 - 2 = 0$$

$$\Rightarrow v = 2t^{\frac{3}{2}}$$

$$s = \int v\,\mathrm{d}t$$

$$s = \int 2t^{\frac{3}{2}}\,\mathrm{d}t$$

$$s = \frac{2}{5} \times 2t^{\frac{5}{2}} + c$$

Since we are interested in a time interval, we do not need to find c.

$$16 = \frac{4}{5} t^{\frac{5}{2}}$$

$$20 = t^{\frac{5}{2}}$$

$$t = 20^{\frac{2}{5}} = 3.3144...$$

The particle takes 3.31 s to travel 16 m.

8 **a** $s = k\sqrt{t}$ m; when $s = 200$ m, $t = 25$ s

$T = 25$ s because the runner completes the race in 25 s.

Also,

$$200 = k\sqrt{25}$$

$$k = \frac{200}{5}$$

$$= 40$$

The values of k and T are 40 and 25 s, respectively.

b $v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$

$$v = \frac{1}{2} \times 40t^{-\frac{1}{2}}$$

$$= \frac{20}{\sqrt{t}}$$

Runner finishes the race in 25 s:

$$v(25) = \frac{20}{\sqrt{25}} = 4$$

The speed of the runner when she crosses the finish line is 4 ms^{-1}.

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8 c For small values of t, v is unrealistically large: For example, at $t = 0.01\text{s}$ $v = 20\times 0.01^{-\frac{1}{2}} = 200\,\text{ms}^{-1}$ and no human could run this fast!

9 a $v = 2 + 8\sin kt$

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = 8k\cos kt$

For any constant k, $\frac{\mathrm{d}}{\mathrm{d}t}(\sin kt) = k\cos kt$

When $t = 0$, $a = 4$

$4 = 8k \Rightarrow k = \frac{1}{2}$

The initial condition, that the acceleration is $4\,\text{m}\,\text{s}^{-1}$, gives an equation in k which you solve.

b $a = 8\times\frac{1}{2}\cos\frac{1}{2}t = 4\cos\frac{1}{2}t$

When $a = 0$

$\cos\frac{1}{2}t = 0$

$\Rightarrow \frac{1}{2}t = \frac{\pi}{2},\ \frac{3\pi}{2}$

$\Rightarrow t = \pi,\ 3\pi$

In all differentiation and integration of trigonometric functions, it is assumed that angles are measured in radians. $\cos\theta = 0$ when θ is an odd multiple of $\frac{\pi}{2}$.

c $64 - (v-2)^2 = 64 - \left(8\sin\frac{1}{2}t\right)^2$

$= 64 - 64\sin^2\frac{1}{2}t$

$= 64\left(1 - \sin^2\frac{1}{2}t\right)$

$= 64\cos^2\frac{1}{2}t$

$= 4\left(4\cos\frac{1}{2}t\right)^2$

$= 4a^2$, as required.

Using the identity $\sin^2\theta + \cos^2\theta = 1$

d At maximum velocity, $a = 0\ \text{ms}^{-2}$
From part **b**, this occurs when $t = \pi$ s and $t = 3\pi$ s.
In both cases, $\sin kt = 1$, so $v = 2 + 8 = 10$.

The maximum value of a occurs when $\frac{\mathrm{d}a}{\mathrm{d}t} = 0$

Since a is a multiple of $\cos kt$, $\frac{\mathrm{d}a}{\mathrm{d}t}$ is a multiple of $\sin kt$,

so $\frac{\mathrm{d}a}{\mathrm{d}t} = 0$ when $\sin kt = 0$ and hence $v = 2 + 0 = 2$ (from **a**)

By the result from **c**, at $v = 2$ we have

$4a^2 = 64 - (2-2)^2$

$a = \sqrt{\frac{64}{4}} = 4$

The maximum values of velocity and acceleration are $10\ \text{ms}^{-1}$ and $4\ \text{ms}^{-2}$ respectively.

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10 a Find acceleration:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}\left(10t - 2t^{\frac{3}{2}}\right)}{\mathrm{d}t}$$

$$a = 10 - 3t^{\frac{1}{2}}$$

For $0 \le t \le 4$, a ≥ 0, so v always increasing and hence maximum value of v occurs at $t = 4$ s.

$$v(4) = (10 \times 4) - \left(2 \times 4^{\frac{3}{2}}\right)$$

$$= 40 - 16$$

$$= 24$$

The maximum velocity for $0 \le t \le 4$ is 24 ms^{-1}

b For first 4 s:

$$s = \int v\,\mathrm{d}t$$

$$s = \int \left(10t - 2t^{\frac{3}{2}}\right)\mathrm{d}t$$

$$s = 5t^2 - \frac{4}{5}t^{\frac{5}{2}} + c$$

At $t = 0$, $s = 0$ so $c = 0$

Hence,

$$s(4) = \left(5 \times 4^2\right) - \left(\frac{4}{5} \times 4^{\frac{5}{2}}\right)$$

$$= 80 - \frac{128}{5}$$

$$= 54.4$$

When $t = 4$ s, P is 54.4 m from O.

c When P is at rest, $v = 0$

$$0 = 24 - \left(\frac{t-4}{2}\right)^4$$

$$\frac{t-4}{2} = \sqrt[4]{24}$$

$$t = \left(2 \times \sqrt[4]{24}\right) + 4 = 8.4267...$$

P is at rest after 8.43 s (3s.f.).

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10 d In first 4 s, *P* travels 54.4 m (see part **b**).
For remaining time:

$$s = \int_4^{10} v \, \mathrm{d}t$$

$$s = \int_4^{8.43} 24 - \left(\frac{t-4}{2}\right)^4 \mathrm{d}t + \left|\int_{8.43}^{10} 24 - \left(\frac{t-4}{2}\right)^2 \mathrm{d}t\right|$$

$$s = \left[24t - \frac{(t-4)^5}{5\times 2^4}\right]_4^{8.43} + \left|\left[24t - \frac{(t-4)^5}{5\times 2^4}\right]_{8.43}^{10}\right|$$

$$s = \left(24\times 8.43 - \frac{4.43^5}{80}\right) - (96-0) + \left|\left(240 - \frac{6^5}{80}\right) - \left(24\times 8.43 - \frac{4.43^5}{80}\right)\right|$$

$$= 85 + |-38.2| - 85 + 38.2 - 123.2$$

P travels a total distance of 54.4 + 123.2 = 177.6 m.