Further kinematics 8B

1 a $\mathbf{u} = (12\mathbf{i} + 24\mathbf{j})\ \text{ms}^{-1}$, $t = 3$ s, $\mathbf{a} = -9.8\mathbf{j}\ \text{ms}^{-2}$, $\mathbf{s} = ?$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = (12\mathbf{i} + 24\mathbf{j}) \times 3 + \frac{1}{2}(-9.8\mathbf{j}) \times 9$$

$$\mathbf{s} = 36\mathbf{i} + 27.9\mathbf{j}$$

The position vector of P after 3 s is $(36\mathbf{i} + 27.9\mathbf{j})$ m.

b $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$\mathbf{v} = (12\mathbf{i} + 24\mathbf{j}) - 3 \times 9.8\mathbf{j}$$

$$\mathbf{v} = 12\mathbf{i} + (24 - 29.4)\mathbf{j}$$

$$\mathbf{v} = 12\mathbf{i} - 5.4\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{12^2 + (-5.4)^2}$$

$$= \sqrt{173.16}$$

$$|\mathbf{v}| = 13.159...$$

The speed of P after 3 s is 13 m s^{-1} (2 s.f.).

2 a $\mathbf{u} = (4\mathbf{i} + 5\mathbf{j})\ \text{ms}^{-1}$, $t = t$ s, $\mathbf{a} = -10\mathbf{j}\ \text{ms}^{-2}$, $\mathbf{s} = ?$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = (4\mathbf{i} + 5\mathbf{j})t + \frac{1}{2}(-10\mathbf{j})t^2$$

$$\mathbf{s} = 4t\mathbf{i} + 5(t - t^2)\mathbf{j}$$

The position vector of the particle after t s is $4\mathbf{i} + 5(t - t^2)\mathbf{j}$ m.

b When particle reaches greatest height, **j** component of velocity = 0 (the **i** component remains unchanged throughout).

$\mathbf{u} = (4\mathbf{i} + 5\mathbf{j})\ \text{ms}^{-1}$, $\mathbf{v} = (4\mathbf{i})\ \text{ms}^{-1}$, $\mathbf{a} = -10\mathbf{j}\ \text{ms}^{-2}$, $t = ?$

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$4\mathbf{i} = 4\mathbf{i} + 5\mathbf{j} - 10t\mathbf{j}$$

$$10t\mathbf{j} = 5\mathbf{j}$$

$$t = 0.5 \text{ s}$$

Using this value of t to determine the coefficient of **j** in the equation derived in part **a**:

$$h = 5(0.5 - 0.5^2)$$

$$h = 5 \times 0.25 = 1.25$$

The greatest height of the particle is 1.25 m.

3 a Both assumptions are made in order to facilitate the calculation. Either could be better or worse than the other. Possible answers include:
The sea is likely to be horizontal and relatively flat, whereas the ball is subject to air resistance, so the assumption that sea is a horizontal plane is most reasonable.
Although the sea is horizontal it is unlikely to be flat because of waves, so the assumption that the ball is a particle is most reasonable.

b $\mathbf{u} = (3p\mathbf{i} + p\mathbf{j})\text{ ms}^{-1}$, $\mathbf{a} = -9.8\mathbf{j}\text{ ms}^{-2}$, $t = 2$ s, $\mathbf{v} = ?$
Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$\mathbf{v} = \begin{pmatrix} 3p \\ p \end{pmatrix} + 2\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} 3p \\ p-19.6 \end{pmatrix} \qquad \textbf{(1)}$$

We also know that $\mathbf{r_0} = 25\mathbf{j}$ m, $\mathbf{r} = (q\mathbf{i} + 10\mathbf{j})$ m.
The change in displacement of the ball is:

$$\mathbf{s} = \mathbf{r} - \mathbf{r_o}$$
$$= q\mathbf{i} + 10\mathbf{j} - 25\mathbf{j}$$
$$\mathbf{s} = q\mathbf{i} - 15\mathbf{j}$$

Using:

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\begin{pmatrix} q \\ -15 \end{pmatrix} = 2\begin{pmatrix} 3p \\ p \end{pmatrix} + \frac{2^2}{2}\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

Comparing $\mathbf{j}$ components:

$$-15 = 2p - 19.6$$
$$2p = 19.6 - 15$$
$$p = \frac{4.6}{2} = 2.3$$

Substitute $p = 2.3$ in (**1**):

$$\mathbf{v} = \begin{pmatrix} 3\times 2.3 \\ 2.3-19.6 \end{pmatrix} = \begin{pmatrix} 6.9 \\ -17.3 \end{pmatrix}$$

The velocity of the ball at B is $(6.9\mathbf{i} - 17\mathbf{j})\text{ ms}^{-1}$ (both coefficients to 2s.f.).

c In order to determine the acceleration on the boat, we first need to find the time at which the ball reaches the sea.
The displacement of the ball relative to A is given by:

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = \begin{pmatrix} 3\times 2.3 \\ 2.3 \end{pmatrix}t + \frac{1}{2}\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}t^2$$

$$\mathbf{s} = \begin{pmatrix} 6.9t \\ 2.3t - 4.9t^2 \end{pmatrix}$$

3 c When the ball lands at C, $\mathbf{s} = x\mathbf{i} - 25\mathbf{j}$, where $x = OC$.

$$\begin{pmatrix} x \\ -25 \end{pmatrix} = \begin{pmatrix} 6.9t \\ 2.3t - 4.9t^2 \end{pmatrix}$$

Considering **j** components only:

$$-25 = 2.3t - 4.9t^2$$

$$4.9t^2 - 2.3t - 25 = 0$$

Finding the positive root of this quadratic equation (negative solution can be ignored as before ball thrown or ship sets out):

$$t = \frac{2.3 \pm \sqrt{2.3^2 + (4 \times 4.9 \times 25)}}{2 \times 4.9}$$

$$t = 2.5056...$$

Considering **i** components only:

$$x = 6.9t = 6.9 \times 2.506 = 17.289 \text{ m}$$

For the boat:

s = 17.289 m, t = 2.506 s, u = 0 ms^{-1}, a = ?

$$s = ut + \frac{1}{2}at^2$$

$$17.289 = 0 + \frac{1}{2}a(2.506)^2$$

$$a = \frac{34.578}{6.28} = 5.50...$$

(solution of $t = 0$ ignored – shows that both boat and ball start at same place)

The acceleration of the boat is 5.5 ms^{-2} (2s.f.).

4 a $\mathbf{u} = (3u\mathbf{i} + 4u\mathbf{j}),\ \mathbf{a} = -9.8\mathbf{j},\ \mathbf{s} = 750\mathbf{i},\ t = t$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$750\mathbf{i} = (3u\mathbf{i} + 4u\mathbf{j})t + \frac{1}{2}(-9.8\mathbf{j})t^2$$

$$750\mathbf{i} = 3ut\mathbf{i} + (4ut - 4.9t^2)\mathbf{j}$$

Comparing **i** coefficients:

$$750 = 3ut$$

$$\therefore t = \frac{250}{u}$$

Comparing **j** coefficients:

$$0 = 4ut - 4.9t^2$$

$$0 = \frac{4u \times 250}{u} - 4.9\left(\frac{250}{u}\right)^2 \quad \text{(substituting } t = \frac{250}{u} \text{ from above)}$$

$$= 1000 - \frac{306\,250}{u^2}$$

$$u^2 = \frac{306\,250}{1000}$$

$$= 306.25$$

$$u = \sqrt{306.25} = 17.5, \text{ as required.}$$

b Greatest height when **j** component of velocity is zero.
Considering **j** components:

$u_y = 4u = 4 \times 17.5 = 70,\ a = -9.8,\ v_y = 0,\ s = ?$

$$v^2 = u^2 + 2as$$

$$0^2 = 70^2 - 2 \times 9.8 \times s$$

$$s = \frac{70^2}{2 \times 9.8}$$

$$= 250$$

P reaches a max height of 250 m above the ground.

c Find the **i** and **j** components of the velocity when $t = 5$, and then find the angle between them.

$\mathbf{u} = (52.5\mathbf{i} + 70\mathbf{j}),\ \mathbf{a} = -9.8\mathbf{j},\ t = 5,\ \mathbf{v} = ?$

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\mathbf{v} = (52.5\mathbf{i} + 70\mathbf{j}) - 5 \times 9.8\mathbf{j}$$

$$\mathbf{v} = 52.5\mathbf{i} - 21\mathbf{j}$$

$$\tan\theta = \frac{v_y}{u_x} = \frac{21}{52.5} = 0.4$$

$$\Rightarrow \theta = 21.8°$$

The angle the direction of motion of *P* makes with **i** when $t = 5$ is $22°$ (to the nearest degree).

5 Let the point S be $x\mathbf{i}+y\mathbf{j}$
$\mathbf{u}=(8\mathbf{i}+10\mathbf{j}),\ \mathbf{a}=-9.8\mathbf{j},\ t=6,\ \mathbf{s}=x\mathbf{i}+y\mathbf{j}$

a Considering **i** components,

$$x=u_x\times t$$
$$=8\times 6$$
$$=48$$

The horizontal distance between O and S is 48 m.

b Considering **j** components,

$$y=ut+\frac{1}{2}at^2$$
$$=10\times 6-4.9\times 6^2$$
$$=-116.4$$

The vertical distance between O and S is 120m (2 s.f.).

c $\mathbf{u}=(8\mathbf{i}+10\mathbf{j}),\ \mathbf{a}=-9.8\mathbf{j},\ t=T,\ \mathbf{v}=8\mathbf{i}-14.5\mathbf{j}$

$$\mathbf{v}=\mathbf{u}+\mathbf{a}t$$
$$8\mathbf{i}-14.5\mathbf{j}=(8\mathbf{i}+10\mathbf{j})-T\times 9.8\mathbf{j}$$

Considering **j** components,

$$-14.5=10-9.8T$$
$$T=\frac{24.5}{9.8}=\frac{5}{2}$$
$$=2\frac{1}{2}$$

At $T=\frac{5}{2}$ s,

$$\mathbf{s}=\mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{s}=\frac{5}{2}(8\mathbf{i}+10\mathbf{j})+\frac{1}{2}(-9.8\mathbf{j})\left(\frac{5}{2}\right)^2$$
$$\mathbf{s}=\left(8\times\frac{5}{2}\right)\mathbf{i}+\left(10\times\frac{5}{2}-4.9\times\left(\frac{5}{2}\right)^2\right)\mathbf{j}$$
$$\mathbf{s}=20\mathbf{i}-\frac{45}{8}\mathbf{j}$$

The position vector of the particle after $2\frac{1}{2}$ seconds is $\left(20\mathbf{i}-\frac{45}{8}\mathbf{j}\right)$ m.

6 a $\mathbf{u} = (a\mathbf{i} + b\mathbf{j})\text{ ms}^{-1}$, $t = t$ s, $\mathbf{a} = -10\mathbf{j}\text{ ms}^{-2}$, $\mathbf{s} = (x\mathbf{i} + y\mathbf{j})$ m

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$x\mathbf{i} + y\mathbf{j} = (a\mathbf{i} + b\mathbf{j})t + \frac{1}{2}(-10\mathbf{j})t^2$$

Considering coefficients of **i**:

$x = at$

$t = \frac{x}{a}$ **(1)**

Considering coefficients of **j**:

$y = bt - 5t^2$ **(2)**

Substituting $t = \frac{x}{a}$ from (**1**) into (**2**):

$y = \frac{bx}{a} - \frac{5x^2}{a^2}$ as required.

b i X is the value of x when y = 0:

$$0 = \frac{bX}{8} - \frac{5X^2}{64}$$

$$\frac{5X^2}{64} = \frac{bX}{8}$$

$5X^2 = 8bX$ disregarding $X = 0$

$$X = \frac{8b}{5}$$

X is $1.6b$

ii Y is the value of y when $x = \frac{X}{2} = \frac{4b}{5}$:

$$Y = \frac{b \times 4b}{8 \times 5} - \frac{5 \times (4b)^2}{64 \times 5^2}$$

$$Y = \frac{b^2}{10} - \frac{b^2}{4 \times 5}$$

$$Y = \frac{b^2}{20}$$

Y is $0.05b^2$