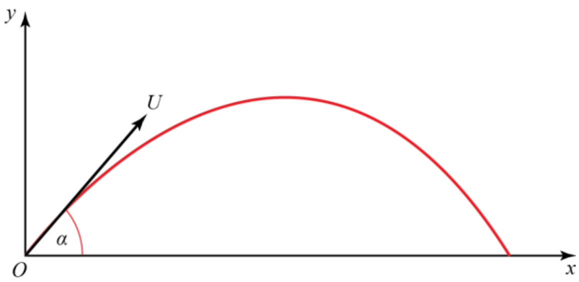
Projectiles 6D



1 At maximum height, h, the vertical component of velocity, $v_y = 0$

R(↑): $u = u_y = U\sin\alpha$, $a = -g$, $s = h$, $v = 0$

$$v^2 = u^2 + 2as$$
$$0 = U^2\sin^2\alpha - 2gh$$
$$2gh = U^2\sin^2\alpha$$
$$h = \frac{U^2\sin^2\alpha}{2g} \quad \text{as required.}$$

2 Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = 21\cos\alpha$

$R(\uparrow)\ u_y = 21\sin\alpha$

a Resolve horizontally and vertically at the point (x, y):

$R(\rightarrow)$

$u = u_x = 21\cos\alpha,\ s = x,\ t = ?$

$$s = ut$$
$$x = t \times 21\cos\alpha$$
$$t = \frac{x}{21\cos\alpha}$$

$R(\uparrow)$

$u = u_y = 21\sin\alpha,\ s = y,\ t = \dfrac{x}{21\cos\alpha},\ a = -g$

$$s = ut + \frac{1}{2}at^2$$
$$y = 21\sin\alpha\left(\frac{x}{21\cos\alpha}\right) - 4.9\left(\frac{x}{21\cos\alpha}\right)^2$$
$$= x\tan\alpha - \frac{4.9x^2}{441\cos^2\alpha}$$
$$= x\tan\alpha - \frac{x^2}{90\cos^2\alpha} \quad \text{as required.}$$

2 b $\frac{1}{\cos^2\alpha} \equiv \sec^2\alpha \equiv 1+\tan^2\alpha$

Hence $\frac{x^2}{90\cos^2\alpha} \equiv \frac{x^2}{90}\left(1+\tan^2\alpha\right)$

Evaluating $y = x\tan\alpha - \frac{x^2}{90\cos^2\alpha}$ when $y = 8.1,\ x = 36$ gives:

$$8.1 = 36\tan\alpha - \frac{36^2}{90}\left(1+\tan^2\alpha\right)$$

$$8.1 = 36\tan\alpha - 14.4\left(1+\tan^2\alpha\right)$$

$$0 = 144\tan^2\alpha - 360\tan\alpha + 225$$

$$0 = 16\tan^2\alpha - 40\tan\alpha + 25$$

$$0 = \left(4\tan\alpha - 5\right)^2$$

$$\frac{5}{4} = \tan\alpha$$

3 Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = U\cos\alpha$

$R(\uparrow)\ u_y = U\sin\alpha$

a We find time of flight by setting $s_y = 0$

$R(\uparrow)$: $s = 0,\ u = U\sin\alpha,\ a = -g,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = Ut\sin\alpha - \frac{1}{2}gt^2$$

$$= t\left(U\sin\alpha - \frac{1}{2}gt\right)$$

$\frac{1}{2}gt = U\sin\alpha$ (ignore $t = 0$, which corresponds to the point of projection)

$t = \frac{2U\sin\alpha}{g}$ as required

b We find range by considering horizontal motion when $t = \frac{2U\sin\alpha}{g}$

$R(\rightarrow)$: $s = R,\ v = U\cos\alpha,\ t = \frac{2U\sin\alpha}{g}$

$$s = vt$$

$$R = U\cos\alpha \times \frac{2U\sin\alpha}{g}$$

$$R = \frac{U^2 \times 2\sin\alpha\cos\alpha}{g}$$

Using the trigonometric identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$, it follows that

$R = \frac{U^2\sin 2\alpha}{g}$, as required

3 **c** The greatest possible value of $\sin 2\alpha$ is 1, which occurs when

$2\alpha = 90°$

$\Rightarrow \alpha = 45°$

Hence, for a fixed *U*, the greatest possible range is when $\alpha = 45°$

d $R = \frac{U^2 \sin 2\alpha}{g} = \frac{2U^2}{5g}$

$\Rightarrow \sin 2\alpha = \frac{2}{5}$

$2\alpha = 23.578°, 156.422°$

$\alpha = 11.79°,\ 78.21°$

The two possible angles of elevation are $12°$ and $78°$ (nearest degree).

4 First find the time it took the firework to reach max. height.

R(↑): initial velocity = *v*, final velocity = 0, $a = -g$, $t = ?$

$v = u + at$

$0 = v - gt$

$t = \frac{v}{g}$

The two parts of the firework will take the same time to fall as the firework did to climb.
Considering the horizontal motion of one part of the firework as it falls:

R(→): $u = 2v$, $t = \frac{v}{g}$, $s = ?$

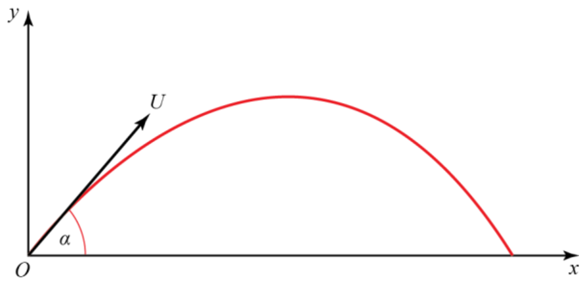
$s = ut$

$s = 2v \times \frac{v}{g}$

$s = \frac{2v^2}{g}$

The other part travels the same distance in the opposite direction, so the two parts land
$\frac{2v^2}{g} + \frac{2v^2}{g} = \frac{4v^2}{g}$ m apart.

5 a Considering horizontal motion, first find time at which $s = x$:



R(→): $u_x = U\cos\alpha,\ s = x,\ t = ?$

$$s = ut$$

$$x = (U\cos\alpha)\times t$$

$$t = \frac{x}{U\cos\alpha}$$

Now consider vertical motion with $t = \dfrac{x}{U\cos\alpha}$ to find y:

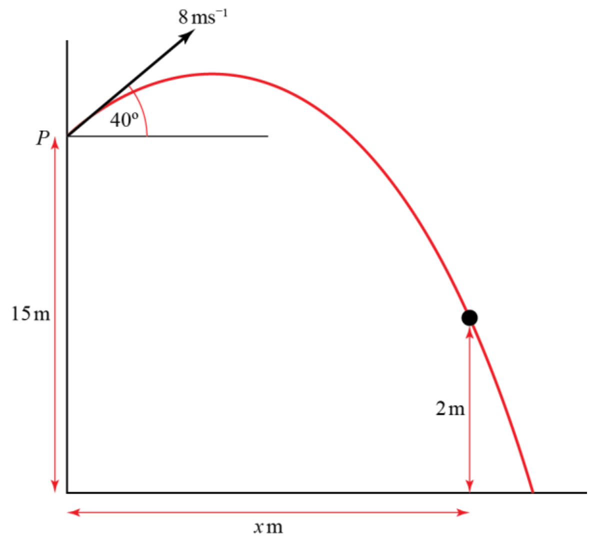
R(↑): $u_y = U\sin\alpha,\ a = -g,\ t = \dfrac{x}{U\cos\alpha},\ s = y$

$$s = ut + \frac{1}{2}at^2$$

$$y = U\sin\alpha\times\frac{x}{U\cos\alpha} - \frac{1}{2}g\left(\frac{x}{U\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{gx^2}{2U^2\cos^2\alpha}$$ as required.

b $U = 8\text{ ms}^{-1}$, $\alpha = 40^\circ$, $y = -13$ m



Substituting these values into the equation derived in **a**:

$$y = x\tan\alpha - \frac{gx^2}{2U^2\cos^2\alpha}$$

$$-13 = x\tan 40^\circ - \frac{9.8x^2}{2\times 8^2\cos^2 40^\circ}$$

$$-13 = 0.8391x - \frac{9.8x^2}{128\times 0.5868}$$

$$-13 = 0.8391x - 0.1305x^2$$

$$0 = 0.1305x^2 - 0.8391x - 13$$

Using the formula for the roots of a quadratic equation:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{0.8391\pm\sqrt{0.8391^2-(4\times 0.1305\times(-13))}}{2\times 0.1305}$$

$$x = \frac{0.8391\pm 2.737}{0.2609}$$

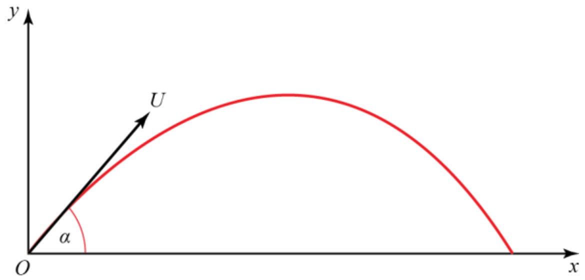
$x = 13.702...$ or $x = -7.2714...$ negative root can be ignored as behind point of projection

The stone is 2 m above sea level at 13.7 m from the end of the pier (to 3 s.f.).

6 a Considering horizontal motion, first find time at which $s = x$:

R(→): $u_x = U\cos\alpha,\ s = x,\ t = ?$

$$s = ut$$
$$x = (U\cos\alpha)\times t$$
$$t = \frac{x}{U\cos\alpha}$$



Now consider vertical motion with $t = \dfrac{x}{U\cos\alpha}$ to find y:

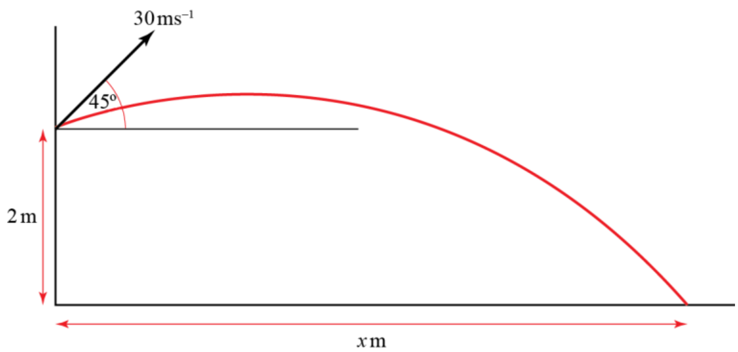
R(↑): $u_y = U\sin\alpha,\ a = -g,\ t = \dfrac{x}{U\cos\alpha},\ s = y$

$$s = ut + \frac{1}{2}at^2$$
$$y = U\sin\alpha\times\frac{x}{U\cos\alpha} - \frac{1}{2}g\left(\frac{x}{U\cos\alpha}\right)^2$$
$$y = x\tan\alpha - \frac{gx^2}{2U^2\cos^2\alpha}$$
$$y = x\tan\alpha - \frac{gx^2}{2U^2}\left(\frac{1}{\cos^2\alpha}\right)$$

but $\dfrac{1}{\cos^2\alpha} \equiv \sec^2\alpha \equiv 1+\tan^2\alpha$ so

$$y = x\tan\alpha - \frac{gx^2}{2U^2}\left(1+\tan^2\alpha\right)$$ as required.

b $U = 30\text{ ms}^{-1}$, $\alpha = 45^\circ$, $y = -2$ m



Substituting these values into the equation derived in **a**:

$$y = x\tan\alpha - \frac{gx^2}{2U^2}\left(1+\tan^2\alpha\right)$$
$$-2 = x\tan 45^\circ - \frac{9.8x^2}{2\times 30^2}\left(1+\tan^2 45^\circ\right)$$
$$-2 = x - \frac{9.8x^2}{900}$$
$$0 = \frac{9.8x^2}{900} - x - 2$$
$$x = \frac{1\pm\sqrt{1^2-(4\times 0.0109\times(-2))}}{2\times 0.0109}$$ (using the quadratic formula)
$$x = \frac{1\pm 1.043}{0.0218}$$

$x = 93.794...$ or $x = -1.9582...$ negative root can be ignored as behind point of projection

The javelin lands 93.8 m from P (to 3s.f.).

6 c As shown in part **a**:

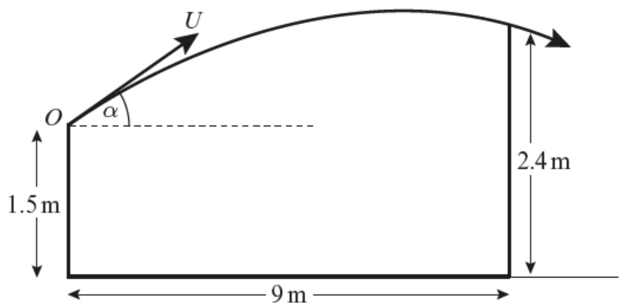
time of flight, $t = \dfrac{x}{U\cos\alpha}$

$U = 30\text{ ms}^{-1}$, $\alpha = 45^{\circ}$, $x = 93.79$ m

$\therefore t = \dfrac{93.79}{30\cos 45^{\circ}} = 4.42$

The javelin lands after 4.4 s.

7 a R(→): $u_x = U\cos\alpha\text{ ms}^{-1}$, $s = 9$ m



$s = vt$

$9 = U\cos\alpha \times t$

$t = \dfrac{9}{U\cos\alpha}$

R(↑): $u_y = U\sin\alpha$, $a = -g$,

$s = 2.4 - 1.5 = 0.9$ m, $t = \dfrac{9}{U\cos\alpha}$

$s = ut + \dfrac{1}{2}at^2$

$0.9 = U\sin\alpha \times \dfrac{9}{U\cos\alpha} - \dfrac{1}{2}g\left(\dfrac{9}{U\cos\alpha}\right)^2$

$0.9 = 9\tan\alpha - \dfrac{81g}{2U^2\cos^2\alpha}$ as required.

b $\alpha = 30^{o} \Rightarrow \tan\alpha = \dfrac{1}{\sqrt{3}}$, $\cos\alpha = \dfrac{\sqrt{3}}{2}$ and $\sin\alpha = \dfrac{1}{2}$

Substituting these values into the equation above:

$0.9 = \dfrac{9}{\sqrt{3}} - \dfrac{4\times 81g}{2U^2\times 3}$

$4.296 = \dfrac{529.2}{U^2}$

$U^2 = \dfrac{529.2}{4.296}$

$U = 11.098...$

When ball passes over the net:

R(→): $v_x = u_x$

$u_x = U\cos 30^{\circ}$

$= 11.10\cos 30^{\circ}$

$= 9.6117...$

R(↑): $u_y = U\sin 30^{\circ}$, $a = -g$, $s = 0.9$ m, $v = ?$

$v^2 = u^2 + 2as$

$v_y^{\,2} = \left(11.10\times\dfrac{1}{2}\right)^2 + 2(-9.8)(0.9)$

$v_y^{\,2} = 30.79 - 17.64 = 13.154...$

7 b The speed at P is given by:

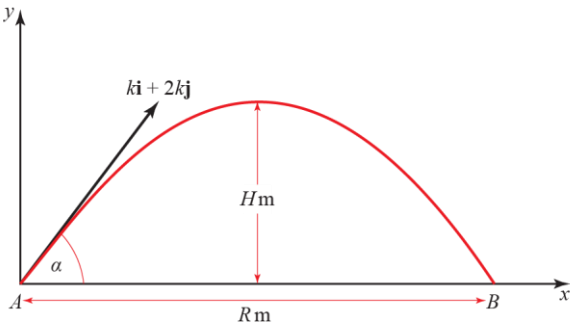
$$v^2 = v_x^2 + v_y^2$$

$$v^2 = 9.612^2 + 13.15$$

$$v = \sqrt{105.5} = 10.273...$$

The ball passes over the net at a speed of 10.3 ms^{-1} (3s.f).

8 a R(→): $u_x = k$ ms^{-1}, $s = x$



$$s = vt$$

$$x = kt$$

$$t = \frac{x}{k}$$

R(↑): $u_y = 2k$ ms^{-1}, $a = -g$, $s = y$, $t = \frac{x}{k}$

$$s = ut + \frac{1}{2}at^2$$

$$y = \frac{2kx}{k} - \frac{1}{2}g\left(\frac{x}{k}\right)^2$$

$$y = 2x - \frac{gx^2}{2k^2} \qquad \text{as required.}$$

b i When $x = R$, $y = 0$

Substituting these values into the equation derived in **a**:

$$0 = 2R - \frac{gR^2}{2k^2}$$

$$\frac{gR^2}{2k^2} = 2R$$

$$R^2 = \frac{2R \times 2k^2}{g}$$

$$R = \frac{4k^2}{g}$$

(The equation also gives a value of $R = 0$. This can be ignored, as it represents the value of x when the object is projected.)

Therefore, the distance AB is $\frac{4k^2}{g}$ m.

8 b ii When $y = H$, $x = \dfrac{R}{2} = \dfrac{2k^2}{g}$

Substituting these values into the equation derived in **a**:

$$H = 2 \times \frac{2k^2}{g} - \frac{g}{2k^2}\left(\frac{2k^2}{g}\right)^2$$

$$H = \frac{4k^2}{g} - \frac{2k^2}{g}$$

$$H = \frac{2k^2}{g}$$

The maximum height reached is $\dfrac{2k^2}{g}$ m.

Challenge

If the point where the stone lands is taken as
$x = x$, $y = 0$,
and stone is projected from a height h m above the hill, then
the equation for the hill is:

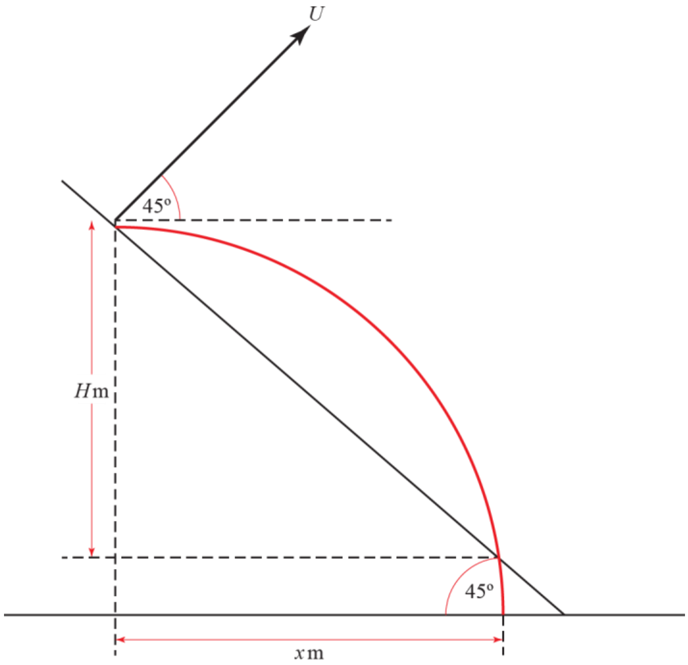
$y = h - x$

and, when $y = 0$

$x = h$

For the stone, $y = -h$

Using the equation for the trajectory of a projectile:



$$y = x\tan\alpha - \frac{gx^2}{2U^2}\left(1+\tan^2\alpha\right)$$

$$-h = x\tan 45^\circ - \frac{gx^2}{2U^2}\left(1+\tan^2 45^\circ\right)$$

$$-h = x - \frac{gx^2}{U^2}$$

But, from above, $x = h$ so:

$$-x = x - \frac{gx^2}{U^2}$$

$$\frac{gx^2}{U^2} = 2x$$

Ignoring the solution $x = 0$:

$$\frac{gx}{U^2} = 2$$

$$x = \frac{2U^2}{g}$$

Therefore, the distance as measured along the slope of the hill, d, is given by:

$$\cos 45^\circ = \frac{x}{d}$$

$$d = \frac{x}{\cos 45^\circ}$$

$$d = \frac{2U^2}{\frac{1}{\sqrt{2}}g} = \frac{2\sqrt{2}U^2}{g}$$ as required.