Review exercise 1

1 a Produce a table for the values of log s and log t:

log s	0.3010	0.6532	0.7924	0.8633	0.9590
log t	−0.4815	0.0607	0.2455	0.3324	0.4698

which produces $r = 0.9992$

b Since r is very close to 1, this indicates that log s by log t is very close to being linear, which means that s and t are related by an equation of the form $t = as^n$ (beginning of Section 1.1).

c Rearranging the equation:

$\log t = -0.9051 + \log s^{1.4437}$

$\Rightarrow t = 10^{-0.9051+\log s^{1.4437}} = 10^{-0.9051} \times 10^{\log s^{1.4437}}$

$\Rightarrow t = 10^{-0.9051} \times s^{1.4437}$

and so $a = 10^{-0.9051} = 0.1244$ (4 s.f.) and $n = 1.4437$

2 a Rearranging the equation:

$y = -0.2139 + 0.0172x$

$\Rightarrow \log t = -0.2139 + 0.0172P$

$\Rightarrow t = 10^{-0.2139+0.0172P} = 10^{-0.2139} \times 10^{0.0172P}$

$\Rightarrow t = 10^{-0.2139} \times \left(10^{0.0172}\right)^P$

Therefore $a = 10^{-0.2139} = 0.611$ (3 s.f.) and $b = 10^{0.0172} = 1.04$ (3 s.f.).

b Not in the range of data (extrapolation).

3 a $r = \dfrac{59.524}{\sqrt{152.444 \times 26.589}}$

$= 0.93494$ (the formulae for this is under S1 in the formula book).

b Make sure your hypotheses are clearly written using the parameter ρ:

$H_0 : \rho = 0, \quad H_1 : \rho > 0$

Test statistic: r = 0.935

Critical value at 1% = 0.7155

(Look up the value under 0.01 in the table for product moment coefficient; quote the figure in full.)

$0.935 > 0.7155$

Draw a conclusion in the context of the question:

So reject H_0 : levels of serum and disease are positively correlated.

4 $r = -0.4063$, critical value for $n = 6$ is -0.6084, so no evidence.

5 a $H_0 : \rho = 0$

$H_1 : \rho < 0$

From the data, $r = -0.9313$. Since the critical value for $n = 5$ is -0.8783, there is sufficient evidence to reject H_0, i.e. at the 2.5% level of significance, there is sufficient evidence to say that there is negative correlation between the number of miles done by a one-year-old car and its value.

b If a 1% level of significance was used, then the critical value for $n = 5$ is -0.9343 and so there would not be sufficient evidence to reject H_0.

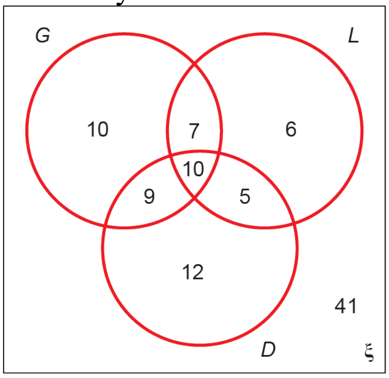
6 a
$$\begin{aligned}\text{P}(\text{tourism}) &= \frac{50}{148}\\ &= \frac{25}{74}\\ &= 0.338 \text{ (3 s.f.)}\end{aligned}$$

b The words ‘given that’ in the question tell you to use conditional probability:
$$\begin{aligned}\text{P}(\text{no glasses} \mid \text{tourism}) &= \frac{\text{P}(G' \cap T)}{\text{P}(T)}\\ &= \frac{\frac{23}{148}}{\frac{50}{148}}\\ &= \frac{23}{50}\\ &= 0.46\end{aligned}$$

c It often helps to write down which combinations you want:
$$\begin{aligned}\text{P}(\text{right-handed}) &= \text{P}(E \cap RH) + \text{P}(T \cap RH) + \text{P}(C \cap RH)\\ &= \frac{30}{148} \times 0.8 + \frac{50}{148} \times 0.7 + \frac{68}{148} \times 0.75\\ &= \frac{55}{74}\\ &= 0.743 \text{ (3 s.f.)}\end{aligned}$$

d The words ‘given that’ in the question tell you to use conditional probability:
$$\begin{aligned}\text{P}(\text{engineering} \mid \text{right-handed}) &= \frac{\text{P}(E \cap RH)}{\text{P}(RH)}\\ &= \frac{\frac{30}{148} \times 0.8}{\frac{55}{74}}\\ &= \frac{12}{55}\\ &= 0.218 \text{ (3 s.f.)}\end{aligned}$$

7 a Start in the middle of the Venn diagram and work outwards. Remember the rectangle and those not in any of the circles. Your numbers should total 100.



b $P(G, L', D') = \frac{10}{100}$

$= \frac{1}{10} = 0.1$

c $P(G', L', D') = \frac{41}{100} = 0.41$

d $P(\text{only two attributes}) = \frac{9+7+5}{100}$

$= \frac{21}{100} = 0.21$

e The word 'given' in the question tells you to use conditional probability:

$P(G \mid L \cap D) = \frac{P(G \mid L \cap D)}{P(L \mid D)}$

$= \frac{\frac{10}{100}}{\frac{15}{100}}$

$= \frac{10}{15}$

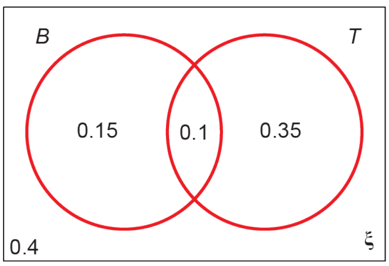
$= \frac{2}{3} = 0.667 \text{ (3 s.f.)}$

8 a $P(B \cup T) = P(B) + P(T) - P(B \cap T)$

$0.6 = 0.25 + 0.45 - P(B \cap T)$

$P(B \cap T) = 0.1$

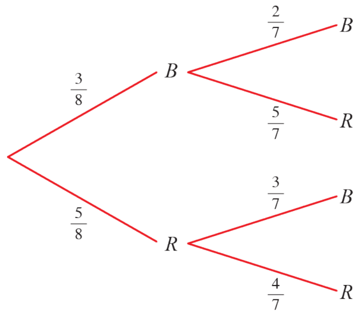
b When drawing the Venn diagram remember to draw a rectangle around the circles and add the probability 0.4.
Remember the total in circle $B = 0.25$ and the total in circle $T = 0.45$.



8 c The words 'given that' in the question tell you to use conditional probability:

$$P(B \cap T' \mid B \cup T) = \frac{0.15}{0.6} = \frac{1}{4} = 0.25$$

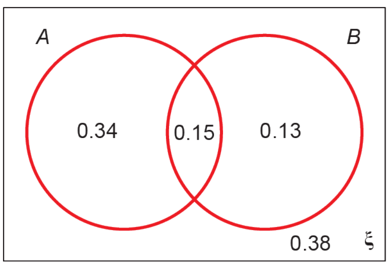
9 a



b i There are two different situations where the second counter drawn is blue. These are BB and RB. Therefore the probability is: $\left(\frac{3}{8}\times\frac{2}{7}\right)+\left(\frac{5}{8}\times\frac{3}{7}\right)=\frac{6+15}{56}=\frac{21}{56}=\frac{3}{8}=0.375$.

ii $\text{P(both blue} \mid \text{2nd blue)} = \frac{\text{P(both blue and 2nd blue)}}{\text{P(2nd blue)}} = \frac{\text{P(both blue)}}{\text{P(2nd blue)}} = \frac{\left(\frac{3}{8}\times\frac{2}{7}\right)}{\left(\frac{3}{8}\right)} = \frac{2}{7}$

10 a The first two probabilities allow two spaces in the Venn diagram to be filled in.
$P(A \cup B) = P(A \cap B') + P(A' \cap B) + P(A \cap B)$, and this can be rearranged to see that $P(A \cap B) = 0.15$
Finally, $P(A \cup B) = 0.62 \Rightarrow P((A \cup B)') = 0.38$. The completed Venn diagram is therefore:



b $P(A) = 0.34 + 0.15 = 0.49$ and $P(B) = 0.13 + 0.15 = 0.28$

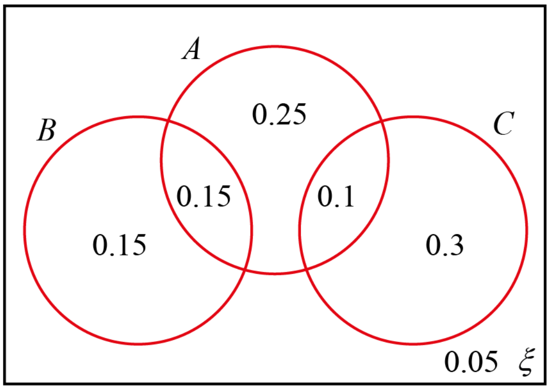
c $P(A \mid B') = \frac{P(A \cap B')}{P(B')} = \frac{0.34}{1 - P(B)} = \frac{0.34}{0.72} = 0.472$ (3 d.p.).

d If A and B are independent, then $P(A) = P(A \mid B) = P(A \mid B')$. From parts **b** and **c**, this is not the case. Therefore they are not independent.

11 a $P(A \cap B) = P(A) \times P(B) \Rightarrow P(A) = P(A \cap B) \div P(B) = 0.15 \div 0.3 = 0.5$

11 b $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.3 - 0.15 = 0.65 \Rightarrow P(A' \cap B') = 1 - 0.65 = 0.35$

c Since B and C are mutually exclusive, they do not intersect.
The intersection of A and C should be 0.1 but $P(A) = 0.5$, allowing $P(A \cap B' \cap C')$ to be calculated. The filled-in probabilities sum to 0.95, and so $P(A' \cap B' \cap C') = 0.05$.
Therefore, the filled-in Venn diagram should look like:



d i $P(A \mid C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{0.1}{0.4} = 0.25$

ii The set $A \cap (B \cup C')$ must be contained within A. First find the set $B \cup C'$: this is made up from four distinct regions on the above Venn diagram, with labels 0.15, 0.15, 0.25 and 0.05. Restricting to those regions that are also contained within A leaves those labelled 0.15 and 0.25. Therefore, $P(A \cap (B \cup C')) = 0.15 + 0.25 = 0.4$

iii From part **ii**, $P(B \cup C') = 0.15 + 0.15 + 0.25 + 0.05 = 0.6$. Therefore

$$P(A \mid (B \cup C')) = \frac{P(A \cap (B \cup C'))}{P(B \cup C')} = \frac{0.4}{0.6} = \frac{2}{3}$$

12 a There are two different events going on: 'Joanna oversleeps' (O) and 'Joanna is late for college' (L). From the context, we cannot assume that these are independent events.
Drawing a Venn diagram, none of the regions can immediately be filled in. We are told that $P(O) = 0.15$ and so $P(J \text{ does not oversleep}) = P(O') = 0.85$. The other two statements can be interpreted as $\dfrac{P(L \cap O)}{P(O)} = 0.75$ and $\dfrac{P(L \cap O')}{P(O')} = 0.1$

Filling in the first one:

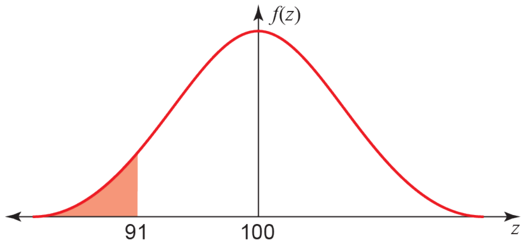
$$\frac{P(L \cap O)}{P(O)} = 0.75 \Rightarrow \frac{P(L \cap O)}{0.15} = 0.75 \Rightarrow P(L \cap O) = 0.1125$$

Also, $\dfrac{P(L \cap O')}{0.85} = 0.1 \Rightarrow P(L \cap O') = 0.085$

Therefore, $P(L) = P(L \cap O) + P(L \cap O') = 0.1125 + 0.085 = 0.1975$

b $P(L \mid O) = \dfrac{P(L \cap O)}{P(O)} = \dfrac{0.1125}{0.1975} = \dfrac{45}{79} = 0.5696$ (4 s.f.).

13 a Drawing a diagram will help you to work out the correct area:

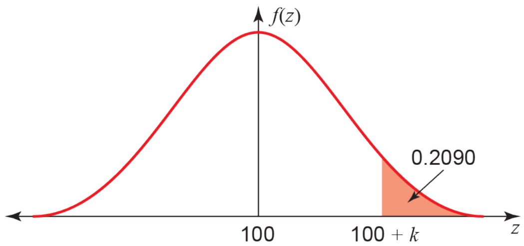


Using $z = \frac{x-\mu}{\sigma}$. As 91 is to the left of 100, your z value should be negative.

$$\begin{aligned}P(X<91) &= P\left(Z<\frac{91-100}{15}\right)\\ &= P(Z<-0.6)\\ &= 1-0.7257\\ &= 0.2743\end{aligned}$$

(The tables give $P(Z<0.6)=P(Z>-0.6)$, so you want $1-$ this probability .)

b



As 0.2090 is not in the table of percentage points you must work out the larger area:
$1-0.2090=0.7910$
Use the first table or calculator to find the z value. It is positive as $100+k$ is to the right of 100.

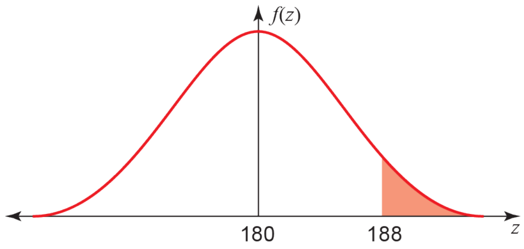
$P(X>100+k)=0.2090$ or $P(X<100+k)=0.791$

$$\frac{100+k-100}{15}=0.81$$

$$k=12$$

14 a Let H be the random variable ~ height of athletes, so $H \sim \text{N}(180,\ 5.2^2)$

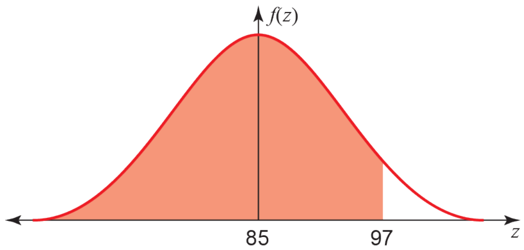
Drawing a diagram will help you to work out the correct area:



Using $z = \dfrac{x-\mu}{\sigma}$. As 188 is to the right of 180 your z value should be positive. The tables give $\text{P}(Z < 1.54)$ so you want $1 -$ this probability:

$$\text{P}(H > 188) = \text{P}\left(Z > \frac{188-100}{5.2}\right)$$
$$= \text{P}(Z > 1.54)$$
$$= 1 - 0.9382$$
$$= 0.0618$$

b Let W be the random variable ~ weight of athletes, so $W \sim \text{N}(85,\ 7.1^2)$



Using $z = \dfrac{x-\mu}{\sigma}$. As 97 is to the right of 85, your z value should be positive.

$$\text{P}(W < 97) = \text{P}\left(Z < \frac{97-85}{7.1}\right)$$
$$= \text{P}(Z < 1.69)$$
$$= 0.9545$$

c $\text{P}(W > 97) = 1 - \text{P}(W < 97)$, so

$$\text{P}(H > 188\ \&\ W > 97) = 0.618(1 - 0.9545)$$
$$= 0.00281$$

d Use the context of the question when you are commenting:

The evidence suggests that height and weight are positively correlated/linked, so assumption of independence is not sensible.

15 a Use the table of percentage points or calculator to find z. You must use at least the four decimal places given in the table.

$P(Z > a) = 0.2$

$a = 0.8416$

$P(Z < b) = 0.3$

$b = -0.5244$

0.5244 is negative since 1.65 is to the left of the centre. 0.8416 is positive as 1.78 is to the right of the centre.

Using $z = \frac{x-\mu}{\sigma}$:

$\frac{1.78-\mu}{\sigma} = 0.8416 \Rightarrow 1.78 - \mu = 0.8416\sigma$ **(1)**

$\frac{1.65-\mu}{\sigma} = -0.5244 \Rightarrow 1.65 - \mu = 0.5244\sigma$ **(2)**

Solving simultaneously, **(1)** – **(2)**:

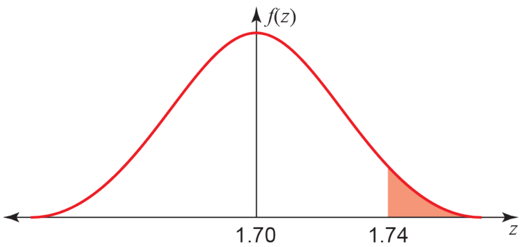
$0.13 = 1.366\sigma$

$\sigma = 0.095$ m

Substitute in **(1)**: $1.78 - \mu = 0.8416 \times 0.095$

$\mu = 1.70$ m

b



Using $z = \frac{x-\mu}{\sigma}$:

$P(\text{height} > 1.74) = P\left(z > \frac{1.74-1.70}{0.095}\right)$

$= P(z > 0.42)$ (the tables give $P(Z < 0.42)$ so you need 1 – this probability)

$= 1 - 0.6628$

$= 0.3372$ (calculator gives 0.3369)

16 a $P(D < 21.5) = 0.32$ and $P(Z < a) = 0.32 \Rightarrow a = -0.467$. Therefore

$\frac{21.5-\mu}{\sigma} = -0.467 \Rightarrow 21.5 - 22 = -0.467\sigma \Rightarrow \sigma = \frac{0.5}{0.467} = 1.071$ (4 s.f.).

b $P(21 < D < 22.5) = P(D < 22.5) - P(D < 21) = 0.5045$ (4 s.f.).

c $P(B \geq 10) = 1 - P(B \leq 9) = 1 - 0.01899 = 0.98101$ (using 4 s.f. for the value given by the binomial distribution) or 0.981 (4 s.f.).

17 a Let W be the random variable 'the number of white plants'. Then $W \sim B(12,\ 0.45)$ ('batches of 12': $n = 12$; '45% have white flowers': $p = 0.45$).

$$P(W = 5) = \binom{12}{5} 0.45^5\, 0.55^7 \text{ (you can also use tables: } P(W \le 5) - P(W \le 4))$$
$$= 0.2225$$

b Batches of 12, so: 7 white, 5 coloured; 8 white, 4 coloured; etc.

$$P(W \ge 7) = 1 - P(W \le 6)$$
$$= 1 - 0.7393$$
$$= 0.2607$$

c Use your answer to part **b**: $p = 0.2607$, $n = 10$:

$$P(\text{exactly } 3) = \binom{10}{3} (0.2607)^3\,(1-0.2607)^7$$
$$= 0.2567$$

d A normal approximation is valid, since n is large (> 50) and p is close to 0.5. Therefore $\mu = np = 150 \times 0.45 = 67.5$ and $\sigma = \sqrt{np(1-p)} = \sqrt{67.5 \times 0.55} = \sqrt{37.125} = 6.093$ (4 s.f.). Now $P(X > 75) \approx P(N > 75.5) = 0.0946$ (3 s.f.).

18 a Using the binomial distribution, $P(B = 35) = \binom{80}{35} \times 0.48^{35} \times 0.52^{45} = 0.06703$.

b A normal approximation is valid, since n is large (> 50) and p is close to 0.5. Therefore $\mu = np = 80 \times 0.48 = 38.4$ and $\sigma = \sqrt{np(1-p)} = \sqrt{38.4 \times 0.52} = \sqrt{19.968} = 4.469$ (4 s.f.). Now $P(B = 35) \approx P(34.5 < N < 35.5) = 0.0668$ (3 s.f.).

Percentage error is $\dfrac{0.06703 - 0.0668}{0.06703} = 0.34\%$.

19 Remember to identify which is H_0 and which is H_1. This is a one-tail test since we are only interested in whether the time taken to solve the puzzle has reduced. You must use the correct parameter (μ):

$H_0 : \mu = 18 \quad H_1 : \mu < 18$

Using $z = \dfrac{x - \mu}{\frac{\sigma}{\sqrt{n}}}$, $z = \dfrac{(16.5 - 18)}{\left(\frac{3}{\sqrt{15}}\right)} = -1.9364...$

Using the percentage point table and quoting the figure in full:
5% one tail c.v. is $z = -1.6449$
$-1.9364 < -1.6449$, so
significant *or* reject H_0 *or* in critical region.
State your conclusion in the context of the question:
There is evidence that the (mean) time to complete the puzzles has reduced.
Or Robert is getting faster (at doing the puzzles).

20 a $P(Z < a) = 0.05 \Rightarrow -1.645$. Using that $P(L < 1.7) = 0.05$ means that
$\frac{1.7 - \mu}{0.4} = -1.645 \Rightarrow 1.7 - \mu = -0.658 \Rightarrow \mu = 2.358$

b $P(L > 2.3) = 0.5576$ (4 s.f.) and so, using the binomial distribution,
$P(B \geq 6) = 1 - P(B \leq 5) = 1 - 0.4758 = 0.5242$ (4 s.f.).

c It is thought that the mean length of the female rattlesnakes is 1.9 m, and a hypothesis test is needed to conclude whether the mean length is not equal to 1.9 m. Therefore,
$H_0 : \mu = 1.9$
$H_1 : \mu \neq 1.9$
Sample size: 20. Therefore, the sample population is initially thought to have distribution
$\bar{M} \sim N\left(1.9, \frac{0.3^2}{20}\right)$. By using the inverse normal distribution, $P(\bar{M} < 1.768) = 0.025$ and
$P(\bar{M} > 2.032) = 0.025$, meaning that the critical region is below 1.768 and above 2.032

d There is sufficient evidence to reject H_0, since 2.09 > 2.032; i.e. there is sufficient evidence to say, at the 5% level, that the mean length of the female rattlesnakes is not equal to 1.9 metres.

21 It is thought that the daily mean temperature in Hurn is less than 12 °C, and so a hypothesis test is needed to conclude whether, at the 5% level of significance, the mean temperature is less than 12 °C. Therefore,
$H_0 : \mu = 12$
$H_1 : \mu < 12$
Sample size: 20. Therefore, the sample population is initially thought to have distribution
$\bar{T} \sim N\left(12, \frac{2.3^2}{20}\right)$. By using the inverse normal distribution, $P(\bar{T} < 11.154) = 0.05$, meaning that the critical region consists of all values below 11.154. Since 11.1 < 11.154, there is sufficient evidence to reject H_0; i.e. there is sufficient evidence to say, at the 5% level, that the mean daily temperature in Hurn is less than 12 °C.

Challenge

1 a Since A and B could be mutually exclusive, $P(A \cap B) \geq 0$. Since $P(A \cap B) \leq P(B) = 0.3$, we have that $0 \leq P(A \cap B) \leq 0.3$ and so $q = P(A \cap B') = P(A) - P(A \cap B)$. Therefore $0.4 \leq p \leq 0.7$

b First, $P(B \cap C) \leq P(B) = 0.3$ and so $q \leq P(B \cap C) - P(A \cap B \cap C) \leq 0.25$. Moreover, it is possible to draw a Venn diagram where $q = 0$, and so $0 \leq q \leq 0.25$

Challenge

2 a We wish to use a hypothesis test to determine (at the 10% significance level) whether the support for the politician is 53%. A normal distribution is suitable, and we use the model given by $\mu = np = 300 \times 0.53 = 159$ and $\sigma = \sqrt{np(1-p)} = \sqrt{159 \times 0.47} = \sqrt{74.73} = 8.645$ (4 s.f.).

Therefore,

$H_0 : \mu = 159$

$H_1 : \mu \neq 159$

By using the inverse normal distribution, $\text{P}(\bar{X} < 144.78) = 0.05$ and $\text{P}(\bar{X} > 173.22) = 0.05$ (2 d.p.) and so the critical region consists of the values below 144.78 and above 173.22

b Since 173 is not within the critical region, there is not sufficient evidence to reject H_0 at the 10% significance level; i.e. there is not sufficient evidence to say, at the 10% level, that the politician's claim that they have support from 53% of the constituents is false.