The normal distribution Mixed exercise 3

1 $H \sim N(178, 4^2)$

a Using the normal CD function, $P(H > 185) = 0.04059... = 0.0401$ (4 d.p.)

b Using the normal CD function, $P(H < 180) = 0.69146...$
The probability that three men, selected at random, all satisfy this criterion is
$P(H < 180)^3 = 0.33060... = 0.3306$ (4 d.p.) .

c Using the inverse normal function, $P(H > h) = 0.005 \Rightarrow h = 188.03...$
To the nearest centimetre, the height of a door frame needs to be at least 188 cm.

2 $W \sim N(32.5,\ 2.2^2)$

a Using the normal CD function, $P(W < 30) = 0.12790...$
The percentage of sheets weighing less than 30kg is 12.8% (3 s.f.).

b Using the normal CD function, $P(31.6 < W < 34.8) = 0.51085...$
So 51.1% of sheets satisfy Bob's requirements.

3 $T \sim N(48, 8^2)$

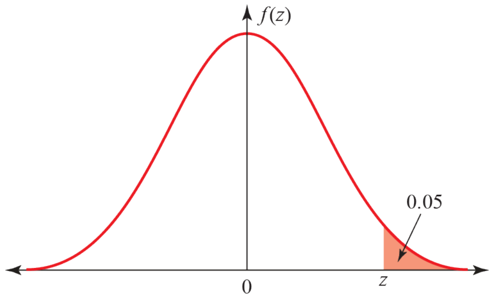
a Using the normal CD function, $P(T > 60) = 0.06680...$
The probability that a battery will last for more than 60 hours is 0.0668 (4 d.p.).

b Using the normal CD function, $P(T < 35) = 0.05208...$
The probability that a battery will last for less than 35 hours is 0.0521 (4 d.p.).

c Use the binomial distribution $X \sim B(30, 0.05208\ldots)$
Using the binomial CD function, $P(X \leqslant 3) = 0.93145...$
The probability that three or fewer last less than 35 hours is 0.9315 (4 d.p.).

4 $X \sim N(24, \sigma^2)$

a $P(X > 30) = 0.05 \Rightarrow P\left(Z > \frac{30 - \mu}{\sigma}\right) = 0.05$



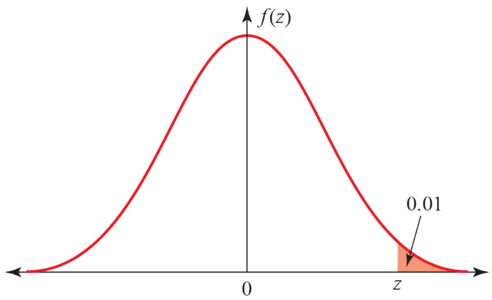
Using the inverse normal function, $z = -1.64485...$

so $1.64485... = \frac{30 - 24}{\sigma}$

$$\sigma = \frac{6}{1.64485...} = 3.647... = 3.65 \text{ (3 s.f.)}$$

b Using the normal CD function, $P(X < 20) = 0.13636... = 0.136$ (3 d.p.)

c $P(X > d) = 0.01 \Rightarrow P\left(Z > \frac{d - \mu}{\sigma}\right) = 0.01$



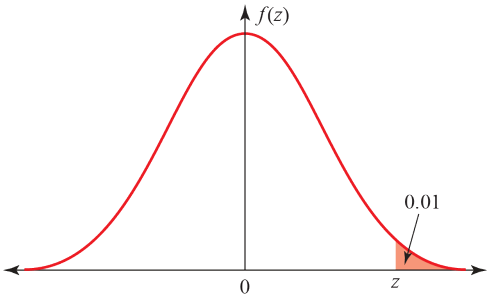
Using the inverse normal function, $z = 2.32634...$

so $2.32634... = \frac{d - 24}{3.647...}$

$$d = 32.485... = 32.5 \text{ (3 s.f.)}$$

5 $L \sim \mathrm{N}(120,\ \sigma^2)$

a $\mathrm{P}(L > 140) = 0.01 \Rightarrow \mathrm{P}\left(Z > \dfrac{140 - \mu}{\sigma}\right) = 0.01$



Using the inverse normal function, $z = 2.32634...$

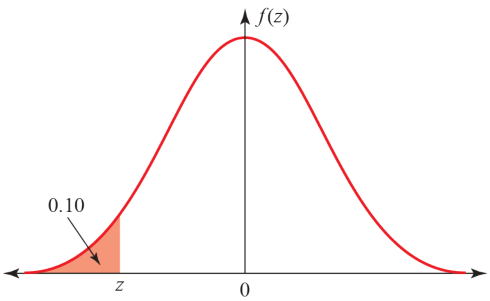
$$\text{so } 2.32634... = \frac{140 - 120}{\sigma}$$

$$\sigma = \frac{20}{2.32634...} = 8.59716...$$

So the standard deviation of the volume dispensed is 8.60 ml (3 s.f.).

b Using the normal CD function, $\mathrm{P}(L < 110) = 0.12237...$
The probability that the machine dispenses less than 110ml is 0.122 (3 s.f.).

c $\mathrm{P}(L < c) = 0.10 \Rightarrow \mathrm{P}\left(Z < \dfrac{c - \mu}{\sigma}\right) = 0.10$



Using the inverse normal function, $z = -1.28155...$

$$\text{so } -1.2816 = \frac{c - 120}{8.59716...}$$

$$c = 108.982...$$

To the nearest millilitre, the largest volume leading to a refund is 109 ml.

6 a $P(X < 20) = 0.25$ and $P(X < 40) = 0.75$

Using the inverse normal function (or the percentage points table),

$$P(X < 20) = 0.25 \Rightarrow P\left(Z < \frac{20-\mu}{\sigma}\right) = 0.25 \Rightarrow z_1 = -0.67448...$$

$$P(X < 40) = 0.75 \Rightarrow P\left(Z < \frac{40-\mu}{\sigma}\right) = 0.75 \Rightarrow z_2 = 0.67448...$$

So $\quad -0.6745\sigma = 20 - \mu \qquad$ **(1)**

and $\quad 0.6745\sigma = 40 - \mu \qquad$ **(2)**

(2) − **(1)**: $1.3489\sigma = 20$

$\sigma = 14.826...$

Substituting into **(2)**:

$\mu = 40 - 0.6745 \times 14.826... = 29.99...$

So $\mu = 30$ and $\sigma = 14.8$ (3 s.f.)

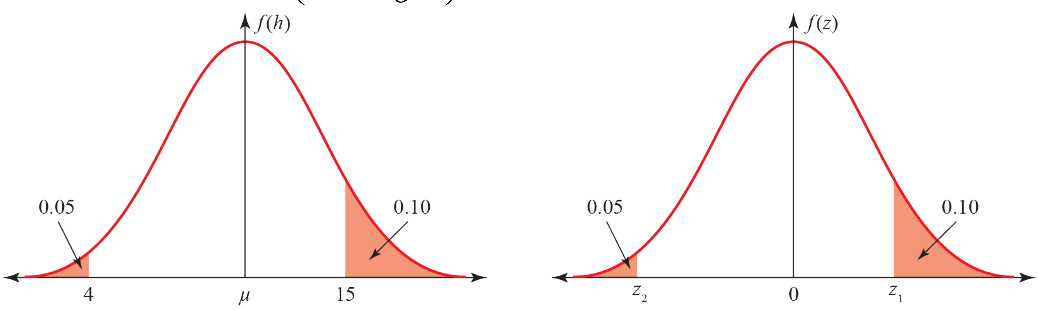
b Using the inverse normal CD function with $\mu = 30$ and $\sigma = 14.826...$,

$P(X < a) = 0.1 \Rightarrow a = 10.999...$ and $P(X < b) = 0.9 \Rightarrow b = 49.000...$

So the 10% to 90% interpercentile range is $49.0 - 11.0 = 38.0$

7 $P(H > 15) = 0.10 \Rightarrow P\left(Z > \frac{15-\mu}{\sigma}\right) = 0.10 \Rightarrow z_1 = 1.28155...$

$P(H < 4) = 0.05 \Rightarrow P\left(Z > \frac{4-\mu}{\sigma}\right) = 0.05 \Rightarrow z_2 = -1.64485...$



So $\quad -1.6449\sigma = 4 - \mu$

$1.2816\sigma = 15 - \mu$

Subtract $\quad 2.9265\sigma = 11$

$\sigma = 3.7587... = 3.76\,\text{cm}$ (3 s.f.)

$\mu = 15 - 1.2816\sigma = 10.2\,\text{cm}$

8 a $T \sim N(80,\ 10^2)$

Using the normal CD function, $P(T > 85) = 0.30853... = 0.3085$ (4 d.p.)

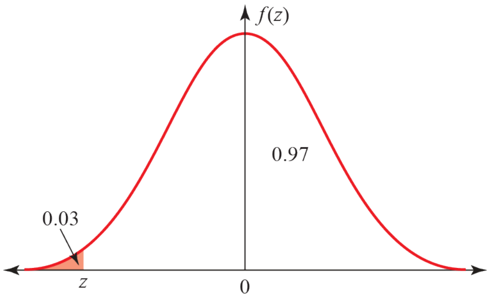
b $S \sim N(100,\ 15^2)$

Using the normal CD function, $P(S > 105) = 0.36944... = 0.3694$ (4 d.p.)

c The student's score on the first test was better, since fewer of the students got this score or higher.

9 $J \sim N(108,\ \sigma^2)$

a $P(J < 100) = 0.03 \Rightarrow P\left(Z < \dfrac{100-\mu}{\sigma}\right) = 0.03$



Using the inverse normal function, $z = -1.88079...$

so $-1.88079... = \dfrac{100-108}{\sigma}$

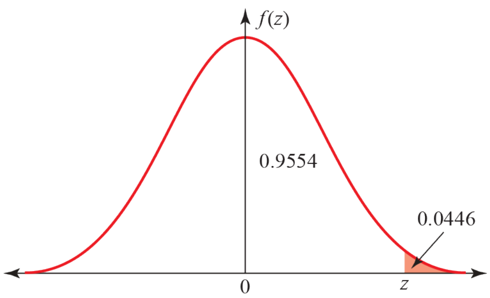
$$\sigma = 4.2535... = 4.25\text{ g } (3\text{ s.f.}).$$

The standard deviation is 4.25 g (3 s.f.).

b Using the normal CD function, $P(J > 115) = 0.0499... = 0.050$ (3 d.p.)

c Use the binomial distribution $X \sim B(25, 0.05)$
Using the binomial CD function,
$P(X \leqslant 2) = 0.87289... = 0.8729$ (4 d.p.)

10 $T \sim N(\mu, 3.8^2)$ and $P(T > 15) = 0.0446$



a $P(T > 15) = 0.0446 \Rightarrow P\left(Z > \dfrac{X-\mu}{\sigma}\right) = 0.0446 \Rightarrow z = 1.70$

so $1.70 = \dfrac{15-\mu}{3.8}$

$$\mu = 15 - 3.8 \times 1.70$$
$$= 8.54 \text{ minutes (3 s.f.)}$$

b $P(T < 5) = P\left(Z < \dfrac{5-8.54}{3.8}\right)$

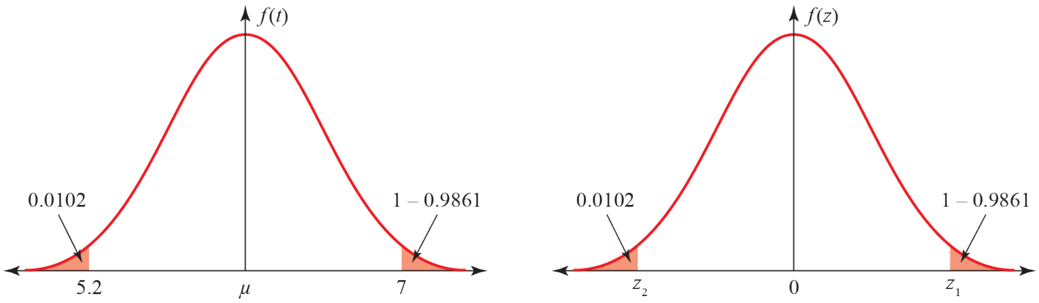
$$= P(Z < -0.93...)$$
$$= 0.17577... = 0.1758 \text{ (4 d.p.)}$$

11 $T \sim \mathrm{N}(\mu, \sigma^2)$

Using the inverse normal function,

$$\mathrm{P}(T < 7) = 0.9861 \Rightarrow \mathrm{P}\left(Z < \frac{7-\mu}{\sigma}\right) = 0.9861 \Rightarrow z_1 = 2.20009...$$

$$\mathrm{P}(T < 5.2) = 0.0102 \Rightarrow \mathrm{P}\left(Z < \frac{5.2-\mu}{\sigma}\right) = 0.0102 \Rightarrow z_2 = -2.31890...$$



So $\quad 2.2001\sigma = 7 - \mu \quad$ **(1)**

and $\quad -2.3189\sigma = 5.2 - \mu \quad$ **(2)**

(1) − **(2)**: $\quad 4.5190\sigma = 1.8$

$$\sigma = 0.3983...$$

Substituting into **(1)**:

$\mu = 7 - 2.2001 \times 0.3983... = 6.123...$

So the mean thickness of the shelving is 6.12 mm and the standard deviation is 0.398 mm (3 s.f.).

12 Let X = number of heads in 60 tosses of a fair coin, so $X \sim \mathrm{B}(60, 0.5)$.

Since $p = 0.5$ and 60 is large, X can be approximated by the normal distribution $Y \sim \mathrm{N}(\mu, \sigma^2)$,

where $\mu = 60 \times 0.5 = 30$ and $\sigma = \sqrt{60 \times 0.5 \times 0.5} = \sqrt{15}$

So $Y \sim \mathrm{N}(30, 15)$

$\mathrm{P}(X < 25) \approx \mathrm{P}(Y < 24.5) = 0.07779... = 0.0778$ (3 s.f.)

13 a The distribution is binomial, $\mathrm{B}(100, 0.40)$.

The binomial distribution can be approximated by the normal distribution when n is large (> 50) and p is close to 0.5. Here $n = 100$ and $p = 0.4$ so both of these conditions are satisfied.

b $\mu = np = 100 \times 0.4 = 40$ and $\sigma = \sqrt{np(1-p)} = \sqrt{40 \times 0.6} = \sqrt{24} = 4.899$ (4 s.f.)

c $\mathrm{P}(X \geqslant 50) \approx \mathrm{P}(Y \geqslant 49.5) = 0.02623... = 0.0262$ (3 s.f.)

14 a $\mathrm{P}(X = 65) = \binom{120}{65} \times 0.46^{65} \times 0.54^{55} = 0.01467$ (4 s.f.) or 0.0147 (4 d.p.)

b The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: $n = 120$ is large (> 50) and $p = 0.46$ is close to 0.5.

$\mu = np = 120 \times 0.46 = 55.2$ and $\sigma = \sqrt{np(1-p)} = \sqrt{55.2 \times 0.54} = \sqrt{29.808} = 5.460$ (4 s.f.)

14 c $Y \sim N(55.2, 5.460^2)$
Using the normal CD function, $P(X = 65) \approx P(64.5 < Y < 65.5) = 0.01463...$
Percentage error $= \dfrac{0.01467 - 0.01463}{0.01467} \times 100 = \dfrac{0.00004}{0.01467} \times 100 = 0.27\%$

15 a The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: $n = 300$ is large (> 50) and $p = 0.6$ is close to 0.5.

b $\mu = np = 300 \times 0.6 = 180$ and $\sigma = \sqrt{np(1-p)} = \sqrt{180 \times 0.4} = \sqrt{72} = 8.485$ (4 s.f.)
So $Y \sim N(180, 8.485^2)$
$P(150 < Y \leqslant 180) \approx P(150.5 < N < 180.5) = 0.52324... = 0.5232$ (4 s.f.)

c Using the inverse normal distribution, $P(N < a) = 0.05 \Rightarrow a = 166.04$
So $P(N < 166.5) > 0.05$ and $P(N < 165.5) < 0.05$
So $165.5 < y < 166.5$, i.e. the smallest value of y such that $P(Y < y) < 0.05$ is $y = 166$.

16 The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: $n = 80$ is large (> 50) and $p = 0.4$ is close to 0.5.
$\mu = np = 80 \times 0.4 = 32$ and $\sigma = \sqrt{np(1-p)} = \sqrt{32 \times 0.6} = \sqrt{19.2} = 4.382$ (4 s.f.)
So $Y \sim N(32, 4.382^2)$
$P(X > 30) \approx P(Y > 30.5) = 0.63394... = 0.6339$ (4 s.f.)

17 a Use the binomial distribution $X \sim B(20, 0.55)$
Using the binomial CD function, $P(X > 10) = 1 - P(X \leqslant 10) = 1 - 0.40863... = 0.5914$ (4 s.f.)

b A normal approximation is valid since $n = 200$ is large (> 50) and $p = 0.55$ is close to 0.5.
$\mu = np = 200 \times 0.55 = 110$ and $\sigma = \sqrt{np(1-p)} = \sqrt{110 \times 0.45} = \sqrt{49.5} = 7.036$ (4 s.f.)
So $Y \sim N(110, 7.036^2)$
$P(X \leqslant 95) \approx P(Y < 95.5) = 0.01965... = 0.0197$ (3 s.f.)

c It seems unlikely that the company's claim is correct: if it were true, the chance of only 95 (or fewer) seedlings producing apples from a sample of 200 seedlings would be less than 2%.

18 a Use the binomial distribution $X \sim B(25, 0.52)$
Using the binomial CD function, $P(X > 12) = 1 - P(X \leqslant 12) = 1 - 0.41992... = 0.5801$ (4 s.f.)

b A normal approximation is valid since $n = 300$ is large (> 50) and $p = 0.52$ is close to 0.5.
$\mu = np = 300 \times 0.52 = 156$ and $\sigma = \sqrt{np(1-p)} = \sqrt{156 \times 0.48} = \sqrt{74.88} = 8.653$ (4 s.f.)
So $Y \sim N(156, 8.653^2)$
$P(X \geqslant 170) \approx P(Y > 169.5) = 0.05936... = 0.0594$ (4 s.f.)

c There is a greater than 5% chance that 170 people out of 300 would be cured, therefore there is insufficient evidence for the herbalist's claim that the new remedy is more effective than the original remedy.

19 $X \sim \text{N}(\mu, 2^2)$

$\text{H}_0 : \mu = 7$, $\text{H}_1 : \mu > 7$, one-tailed test at the 5% level.

Assume H_0, so that $X \sim \text{N}(7, 2^2)$ and $\overline{X} \sim \left(7, \frac{2^2}{25}\right)$

Let $Z = \dfrac{\overline{X} - 7}{\frac{2}{\sqrt{25}}}$

Using the inverse normal function, $\text{P}(Z > z) = 0.05 \Rightarrow z = 1.6449$

$1.6449 = \dfrac{\overline{X} - 7}{\frac{2}{5}} \Rightarrow \overline{X} = 7 + 1.6449 \times \dfrac{2}{5} = 7.6579...$

So the critical region is $\overline{X} > 7.6579...$ or $7.66\,\text{cm}$ (3 s.f.).

20 Let B represent the amount of water in a bottle, so $B \sim \text{N}(\mu, 2^2)$.

$\text{H}_0 : \mu = 125$, $\text{H}_1 : \mu < 125$, one-tailed test at the 5% level.

Assume H_0, so that $B \sim \text{N}(125, 2^2)$ and $\overline{B} \sim \left(125, \frac{2^2}{15}\right)$

Using the normal CD function, $\text{P}(\overline{B} < 124.2) = 0.06066... = 0.0607$ (3 s.f.)

$0.0607 > 0.05$ so not significant, so accept H_0.

There is insufficient evidence to conclude that the mean content of a bottle is lower than the manufacturer's claim.

21 Let B represent the breaking strength, so $B \sim \text{N}(170.2, 10.5^2)$.

a Using the normal CD function, $\text{P}(174.5 < B < 175.5) = 0.03421... = 0.0342$ (3 s.f.)

b $n = 50$ so $\overline{B} \sim \text{N}\left(170.2, \dfrac{10.5^2}{50}\right)$

Using the normal CD function, $\text{P}(\overline{B} > 172.4) = 0.06922... = 0.0692$ (3 s.f.)

c $\text{H}_0 : \mu = 170.2$, $\text{H}_1 : \mu > 170.2$ one-tailed test at the 5% level.

Assume H_0, so that $B \sim \text{N}(170.2, 10.5^2)$ and $\overline{B} \sim \left(170.2, \dfrac{10.5^2}{50}\right)$ (as before).

Using the normal CD function, $\text{P}(\overline{B} > 172.4) = 0.06922... = 0.0692$ (3 s.f.)

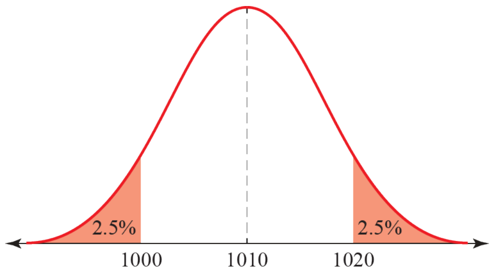
This is the p-value for the hypothesis test.

$0.0692 > 0.05$ so not significant, so accept H_0.

There is insufficient evidence to conclude that the mean breaking strength is increased.

22 Let W represent the weight of sugar in a packet, so $W \sim \text{N}(1010, \sigma^2)$.

a $\text{P}(1000 < W < 1020) = 0.95 \Rightarrow \text{P}(W < 1000) = 0.025 \Rightarrow \text{P}\left(Z < \frac{1000-1010}{\sigma}\right) = 0.025$



Using the inverse normal function, $z = -1.95996...$

so $\quad -1.95996... = \frac{1000-1010}{\sigma}$

$$\sigma = \frac{-10}{-1.95996..} = 5.1021...$$

$$\sigma^2 = 26.031... = 26.03 \text{ (2 d.p.)}$$

b $n = 8$ and $\sum x = 8109.1$, so $\bar{x} = 1013.6375$

$\text{H}_0 : \mu = 1010,\ \text{H}_1 : \mu \neq 1010$, two-tailed test with 1% in each tail.

Assume H_0, so that $W \sim \text{N}(1010, 26.03)$ and $\overline{W} \sim \left(1010, \frac{26.03}{8}\right)$

Using the normal CD function, $\text{P}(\overline{W} > 1013.6375) = 0.02187... = 0.0219$ (3 s.f.)

$0.0219 > 0.01$ so not significant, so accept H_0.

There is insufficient evidence of a deviation in the mean from 1010, so we can assume that condition **i** is being met.

23 Let D represent the diameter of a little-gull egg, so $D \sim \text{N}(4.11, 0.19^2)$.

a Using the normal CD function, $\text{P}(3.9 < D < 4.5) = 0.84542... = 0.8454$ (4 s.f.)

b $\sigma = 0.19,\ n = 8,\ \sum d = 34.5,\ \bar{d} = 4.3125$

$\text{H}_0 : \mu = 4.11,\ \text{H}_1 : \mu \neq 4.11$, two-tailed test with 0.5% in each tail.

Assume H_0, so that $D \sim \text{N}(4.11, 0.19^2)$ and $\overline{D} \sim \left(4.11, \frac{0.19^2}{8}\right)$

Using the normal CD function, $\text{P}(\overline{D} > 4.3125) = 0.00128... = 0.0013$ (2 s.f.)

p-value $= 2 \times 0.00128\ldots = 0.00258 < 0.01$ so significant, so reject H_0.

There is evidence that the mean diameter of eggs from this island is different from elsewhere.

24 **a** $X \sim \text{N}(\mu, \sigma^2)$

$$\overline{X} \sim \text{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

24 b $\mathrm{P}\left(\left|\overline{X}-\mu\right|<15\right)=\mathrm{P}\left(|Z|<\dfrac{15}{\left(\frac{\sigma}{\sqrt{40}}\right)}\right)$

Require $\mathrm{P}\left(|Z|<\dfrac{15\sqrt{n}}{40}\right)>0.95 \Rightarrow \mathrm{P}\left(Z<\dfrac{15\sqrt{n}}{40}\right)<0.025$ (by symmetry)

Using the inverse normal function, $z=-1.95996...$

so $\dfrac{15\sqrt{n}}{40}>1.95996...$

$\sqrt{n}>\dfrac{40\times 1.95996...}{15}=5.2266...$

$n>27.317...$

So a sample of at least 28 is needed.

Challenge

a Use the binomial distribution $X \sim \mathrm{B}(15, 0.48)$.
Using the binomial CD function, $\mathrm{P}(X>8)=1-\mathrm{P}(X\leqslant 8)=1-0.74903...=0.2510$ (4 s.f.)

b The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: $n=250$ is large (> 50) and $p=0.48$ is close to 0.5.

$\mu=np=250\times 0.48=120$ and $\sigma=\sqrt{np(1-p)}=\sqrt{120\times 0.52}=\sqrt{62.4}=7.90$ (3 s.f.)

Test to determine whether the mean is different from 120 (the expected number of supporters based on the manager's claim):

$\mathrm{H}_0: \mu=120$, $\mathrm{H}_1: \mu\neq 120$, two-tailed test with 2.5% in each tail.

Assume H_0, so that $X\sim \mathrm{N}(120, 7.9^2)$ and $\overline{X}\sim\left(120, \dfrac{7.9^2}{250}\right)$

Using the inverse normal function,

$\mathrm{P}(\overline{X}<\overline{x})=0.025 \Rightarrow \overline{x}=104.516$ and $\mathrm{P}(\overline{X}>\overline{x})=0.025\Rightarrow \overline{x}=135.484$

So the critical region approximated for the binomial distribution is $\overline{X}\leqslant 105$ or $\overline{X}\geqslant 135$.

(Note: 105 lies in the critical region because, as part of the normal distribution, it includes the region between 104.5 and 105.5 and 104.513 is where the critical region starts. Similarly 135 lies in the critical region because, as part of the normal distribution, it includes the region between 134.5 and 135.5 and 135.487 is where the critical region starts.)

c Since $102<105$, there is sufficient evidence to reject H_0, i.e. there is sufficient evidence to say, at the 5% level, that the level of support for the manager is different from 48%.