The normal distribution 3G

1 a $H_0 : \mu = 21$, $H_1 : \mu \neq 21$, two-tailed test with 2.5% in each tail.

Assume H_0, so that $X \sim N(21, 1.5^2)$ and $\overline{X} \sim \left(21, \frac{1.5^2}{20}\right)$ or $\overline{X} \sim (21, 0.0.3354^2)$

Using the cumulative normal function, $P(\overline{X} > 21.2) = 0.2755$ (4 d.p.)
$0.2755 > 0.025$ so not significant. Do not reject H_0.

b $H_0 : \mu = 100$, $H_1 : \mu < 100$, one-tailed test at the 5% level.

Assume H_0, so that $X \sim N(100, 5^2)$ and $\overline{X} \sim \left(100, \frac{5^2}{36}\right)$ or $\overline{X} \sim (100, 0.8333^2)$

$P(\overline{X} < 98.5) = 0.0359$ (4 d.p.)
$0.0359 < 0.05$ so significant. Do not reject H_0.

c $H_0 : \mu = 5$, $H_1 : \mu \neq 5$, two-tailed test with 2.5% in each tail.

Assume H_0, so that $X \sim N(5, 3^2)$ and $\overline{X} \sim \left(5, \frac{3^2}{25}\right)$ or $\overline{X} \sim (5, 0.6^2)$

$P(\overline{X} > 6.1) = 0.0334$ (4 d.p.)
$0.0334 > 0.025$ so not significant. Do not reject H_0.

d $H_0 : \mu = 15$, $H_1 : \mu > 15$, one-tailed test at the 1% level.

Assume H_0, so that $X \sim N(15, 3.5^2)$ and $\overline{X} \sim \left(15, \frac{3.5^2}{40}\right)$ or $\overline{X} \sim (5, 0.5534^2)$

$P(\overline{X} > 16.5) = 0.0034$ (4 d.p.)
$0.0034 < 0.01$ so significant. Reject H_0.

e $H_0 : \mu = 50$, $H_1 : \mu \neq 50$, two-tailed test with 0.5% in each tail.

Assume H_0, so that $X \sim N(50, 4^2)$ and $\overline{X} \sim \left(50, \frac{4^2}{60}\right)$ or $\overline{X} \sim (50, 0.5163^2)$

$P(\overline{X} < 48.9) = 0.0166$ (4 d.p.)
$0.0166 > 0.005$ so not significant. Do not reject H_0.

2 a $H_0 : \mu = 120$, $H_1 : \mu < 120$, one-tailed test at the 5% level.

Assume H_0, so that $X \sim N(120, 2^2)$ and $\overline{X} \sim \left(120, \frac{2^2}{30}\right)$

Let $Z = \dfrac{\overline{X} - 120}{\frac{2}{\sqrt{30}}}$

Using the inverse normal function, $P(Z < z) = 0.05 \Rightarrow z = -1.6449$

$$-1.6449 = \frac{\overline{X} - 120}{\frac{2}{\sqrt{30}}} \Rightarrow \overline{X} = 120 - 1.6449 \times \frac{2}{\sqrt{30}} = 119.39...$$

So the critical region is $\overline{X} < 119.39...$ or 119 (3 s.f.).

2 b $H_0 : \mu = 12.5$, $H_1 : \mu > 12.5$, one-tailed test at the 1% level.

Assume H_0, so that $X \sim N(12.5, 1.5^2)$ and $\overline{X} \sim \left(12.5, \frac{1.5^2}{25}\right)$

Let $Z = \frac{\overline{X} - 12.5}{\frac{1.5}{\sqrt{25}}} = \frac{\overline{X} - 12.5}{\frac{1.5}{5}}$

$P(Z > z) = 0.01 \Rightarrow z = 2.3263$

$2.3263 = \frac{\overline{X} - 12.5}{\frac{1.5}{5}} \Rightarrow \overline{X} = 12.5 + 2.3263 \times \frac{3}{10} = 13.197...$

So the critical region is $\overline{X} > 13.197...$ or 13.2 (3 s.f.) (3 s.f.).

c $H_0 : \mu = 85$, $H_1 : \mu > 85$, one-tailed test at the 10% level.

Assume H_0, so that $X \sim N(85, 4^2)$ and $\overline{X} \sim \left(85, \frac{4^2}{50}\right)$

Let $Z = \frac{\overline{X} - 85}{\frac{4}{\sqrt{50}}}$

$P(Z < z) = 0.10 \Rightarrow z = -1.2816$

$-1.2816 = \frac{\overline{X} - 85}{\frac{4}{\sqrt{50}}} \Rightarrow \overline{X} = 85 - 1.2816 \times \frac{4}{\sqrt{50}} = 84.275...$

So the critical region is $\overline{X} < 84.275...$ or 84.3 (3 s.f.).

d $H_0 : \mu = 0$, $H_1 : \mu \neq 0$, two-tailed test with 2.5% in each tail.

Assume H_0, so that $X \sim N(0, 3^2)$ and $\overline{X} \sim \left(0, \frac{3^2}{45}\right)$

Let $Z = \frac{\overline{X} - 0}{\frac{3}{\sqrt{45}}}$

$P(Z < z) = 0.025 \Rightarrow z = -1.9600 \Rightarrow \overline{X} = -1.96 \times \frac{3}{\sqrt{45}} = -0.8765...$

$P(Z > z) = 0.025 \Rightarrow z = 1.9600 \Rightarrow \overline{X} = 1.96 \times \frac{3}{\sqrt{45}} = 0.8765...$

So the critical region is $\overline{X} < -0.877$ or $\overline{X} > 0.877$ (3 s.f.).

e $H_0 : \mu = -8$, $H_1 : \mu \neq -8$, two-tailed test with 0.5% in each tail.

Assume H_0, so that $X \sim N(-8, 1.2^2)$ and $\overline{X} \sim \left(-8, \frac{1.2^2}{20}\right)$

Let $Z = \frac{\overline{X} - (-8)}{\frac{1.2}{\sqrt{20}}}$

$P(Z < z) = 0.005 \Rightarrow z = -2.5758 \Rightarrow \overline{X} = -8 - 2.5758 \times \frac{1.2}{\sqrt{20}} = -8.6911...$

$P(Z > z) = 0.005 \Rightarrow z = 2.5758 \Rightarrow \overline{X} = -8 + 2.5758 \times \frac{1.2}{\sqrt{20}} = -7.3088...$

So the critical region is $\overline{X} < -8.69$ or $\overline{X} > -7.31$ (3 s.f.).

3 $\sigma = 15,\ n = 25,\ \bar{x} = 179$

$H_0 : \mu = 185$ (no improvement), $H_1 : \mu < 185$ (shorter time), one-tailed test at the 5% level.

Assume H_0, so that $X \sim N(185, 15^2)$ and $\bar{X} \sim \left(15, \frac{15^2}{25}\right)$ or $\bar{X} \sim (15, 3^2)$

Using the cumulative normal function, $P(\bar{X} < 179) = 0.0227$ (4 d.p.)

$0.0227 < 0.05$ so significant, so reject H_0.

There is evidence that the new formula is an improvement.

4 a The psychologist wishes to test whether the score has increased (not just changed). Therefore $H_0 : \mu = 100,\ \ H_1 : \mu > 100$, one-tailed test at the 2.5% level.

Assume H_0, so that $X \sim N(100, 15^2)$ and $\bar{X} \sim \left(100, \frac{15^2}{80}\right)$ or $\bar{X} \sim (100, 1.6771^2)$

Using the inverse normal function, $P(\bar{X} > \bar{x}) = 0.025 \Rightarrow \bar{x} = 103.287\ldots$

So the critical region is $\bar{X} > 103.287...$ or 103.29 (3 s.f.).

b $102.5 < 103.29$ so there is not sufficient evidence to reject H_0, i.e. there is not sufficient evidence to say, at the 2.5% level, that eating chocolate before taking an IQ test improves the result.

5 a $\sigma = 0.15,\ n = 30,\ \bar{x} = 8.95$

$H_0 : \mu = 9$ (no change), $H_1 : \mu \neq 9$ (change in mean diameter)

Two-tailed test with 2.5% in each tail

Assume H_0, so that $X \sim N(9.0, 0.15^2)$ and $\bar{X} \sim \left(9.0, \frac{0.15^2}{30}\right)$ or $\bar{X} \sim (9.0, 0.0274^2)$

Using the cumulative normal function, $P(\bar{X} < 8.95) = 0.033944... = 0.0340$ (4 d.p.)

$0.0340 > 0.025$ so not significant, so do not reject H_0.

There is not enough evidence to conclude that there has been a change in the mean diameter.

b Two-tailed test so double to probability to find the p-value

p-value = $0.033944\ldots \times 2 = 0.0678$ (4 d.p.)

6 a First find the mean of the distribution.

$P(D > 5.62) = 0.05$

Using the inverse normal function (or the percentage points table), $p = 0.05 \Rightarrow z = 1.6449$

Using the formula $z = \frac{x - \mu}{\sigma}$, $\frac{5.62 - \mu}{0.1} = 1.6449 \Rightarrow 5.62 - \mu = 0.16449 \Rightarrow \mu = 5.4555$

The probability that a randomly chosen bolt can be sold is $P(5.1 \leqslant D \leqslant 5.6)$

Using the cumulative normal function, $P(5.1 \leqslant D \leqslant 5.6) = 0.92558...$

So the probability that a randomly chosen bolt can be sold is 0.9256 (4 d.p.).

b Use the binomial distribution $N \sim B(12, 1 - 0.9256)$ or $N \sim B(12, 1 - 0.0744)$.

Using the cumulative binomial function, $P(N < 3) = P(N \leqslant 2) = 0.94549...$

So the probability that fewer than three cannot be sold is 0.9455 (4 d.p.).

6 c Test to determine whether the mean diameter is less than 5.7 mm. Therefore
$H_0 : \mu = 5.7$, $H_1 : \mu < 5.7$, one-tailed test at the 2.5% level.

Assume H_0, so that $Y \sim N(5.7, 0.08^2)$ and $\overline{Y} \sim \left(5.7, \frac{0.08^2}{10}\right)$ or $\overline{Y} \sim (5.7, 0.025298^2)$

Using the cumulative normal function, $P(\overline{Y} < 5.65) = 0.02405... < 0.025$ (one-tailed)
so reject H_0.

There is sufficient evidence to suggest that the mean diameter is less than 5.7 mm.

7 a $P(M > 160) = 0.025$
Using the inverse normal function (or the percentage points table), $p = 0.025 \Rightarrow z = 1.9599$

Using the formula $z = \frac{x - \mu}{\sigma}$, $\frac{160 - \mu}{12} = 1.9599 \Rightarrow 160 - \mu = 23.52 \Rightarrow \mu = 136.48$
So the mean mass of a European water vole is 136.48 g (2 d.p.).

b Using the cumulative normal function, $P(M > 150) = 0.1299$
Use the binomial distribution $N \sim B(8, 0.1299)$.
Using the cumulative binomial function, $P(N \geqslant 4) = 1 - P(N \leqslant 3) = 1 - 0.98708... = 0.01291...$
The probability that at least 4 voles have a mass greater than 150 g is 0.0129 (4 d.p.).

7 c Test to determine whether the mean mass is different from 860 grams. Therefore
$H_0 : \mu = 860$, $H_1 : \mu \neq 860$, two-tailed test with 5% in each tail.

Assume H_0, so that $N \sim N(860, 85^2)$ and $\overline{N} \sim \left(860, \frac{85^2}{15}\right)$ or $\overline{N} \sim (860, 21.946^2)$

Using the cumulative normal function, $P(\overline{N} > 875) = 0.24715...$
$0.24715 > 0.05$ so not significant, do not reject H_0.
There is insufficient evidence to suggest that the mean mass of all water rats is different from 860 g.

8 Test to determine whether the daily mean windspeed is greater than 9.5 knots. Therefore
$H_0 : \mu = 9.5$, $H_1 : \mu > 9.5$, one-tailed test at the 2.5% level.

Assume H_0, so that $X \sim N(9.5, 3.1^2)$ and $\overline{X} \sim \left(9.5, \frac{3.1^2}{25}\right)$ or $\overline{X} \sim (9.5, 0.62^2)$

Using the inverse normal function, $P(\overline{X} > \overline{x}) = 0.025 \Rightarrow \overline{x} = 10.715...$

So the critical region is $\overline{X} \geqslant 10.715...$
$12.2 > 10.715$ so there is sufficient evidence to reject H_0, i.e. there is sufficient evidence to say, at the 2.5% level, that the daily mean windspeed is greater than 9.5 knots.