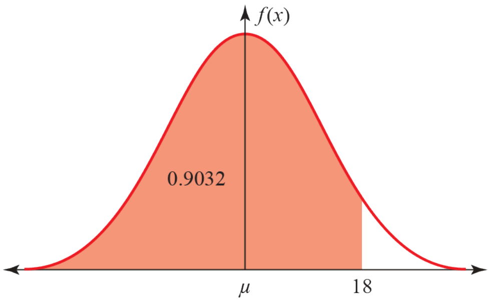
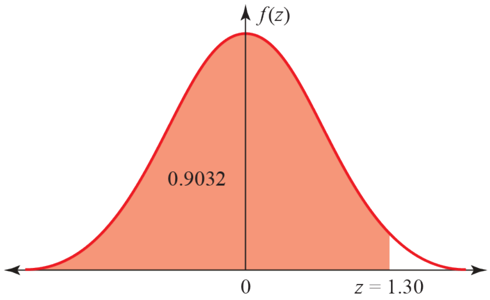
The normal distribution 3E

1 $P(X < 18) = 0.9032 \Rightarrow P\left(Z < \frac{18-\mu}{5}\right) = 0.9032$



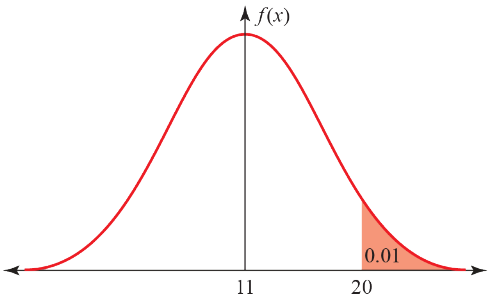


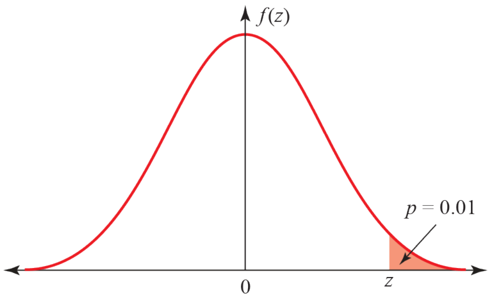
Using the inverse normal function, $z = 1.30$

so $1.30 = \frac{18-\mu}{5}$

$\mu = 18 - 5 \times 1.30 = 11.5$

2 $\text{P}(X > 20) = 0.01 \Rightarrow \text{P}\left(Z < \frac{20-11}{\sigma}\right) = 0.01$





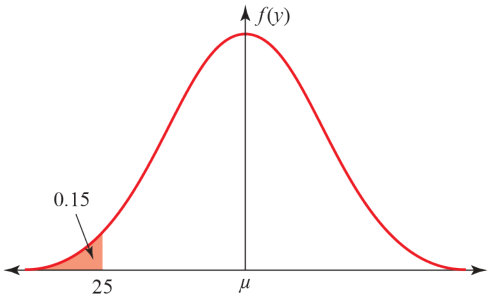
Using the inverse normal function, $z = 2.3263...$

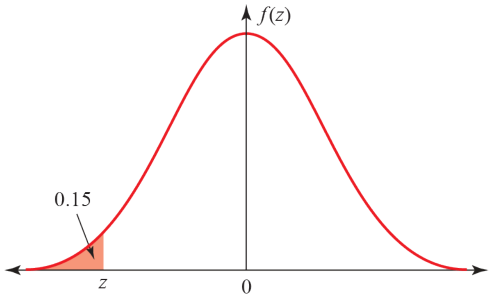
so $2.3263... = \frac{20-11}{\sigma}$

$\sigma = \frac{9}{2.3263...}$

$= 3.8687... = 3.87$ (3 s.f.)

3 $P(Y < 25) = 0.15 \Rightarrow P\left(Z < \frac{25-\mu}{\sqrt{40}}\right) = 0.15$





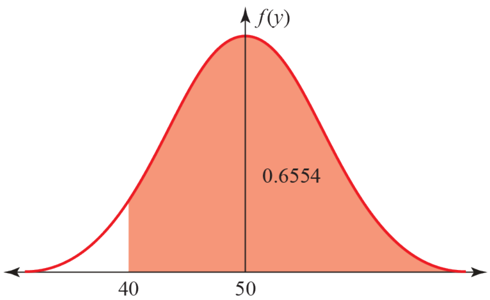
Using the inverse normal function, $z = -1.0364...$

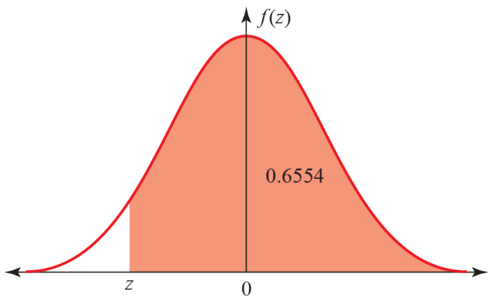
so $-1.0364... = \frac{25-\mu}{\sqrt{40}}$

$$\mu = \sqrt{40} \times (-1.0364...)$$

$$= 31.554... = 31.6 \text{ (3 s.f.)}$$

4 $P(Y > 40) = 0.6554 \Rightarrow P\left(Z > \frac{40-50}{\sigma}\right) = 0.6554$





Using the inverse normal function, $z = -0.3999...$

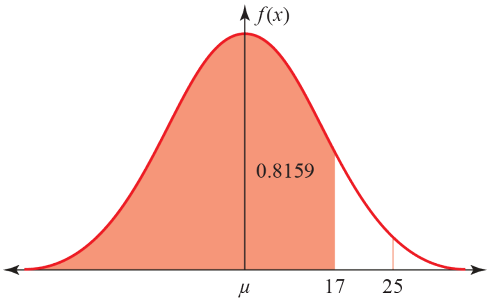
so $-0.3999... = \frac{40-50}{\sigma}$

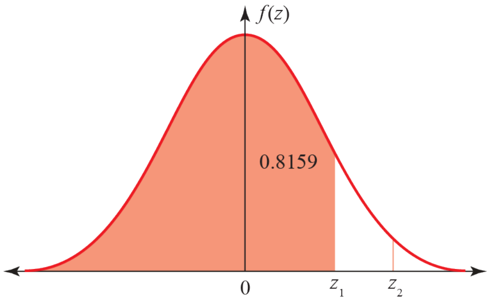
$$\sigma = \frac{10}{0.3999...} = 25.0 \text{ (3 s.f.)}$$

5 Using the inverse normal function,

$$P(X < 17) = 0.8159 \Rightarrow P\left(Z < \frac{17-\mu}{\sigma}\right) = 0.8159 \Rightarrow z_1 = 0.8998...$$

$$P(X < 25) = 0.9970 \Rightarrow P\left(Z < \frac{25-\mu}{\sigma}\right) = 0.9970 \Rightarrow z_2 = 2.7477...$$





So $0.8998\sigma = 17 - \mu$ **(1)**

and $2.7477\sigma = 25 - \mu$ **(2)**

(2) – **(1)**: $1.8479\sigma = 8$

$\sigma = 4.329...$

Substituting into (**2**):

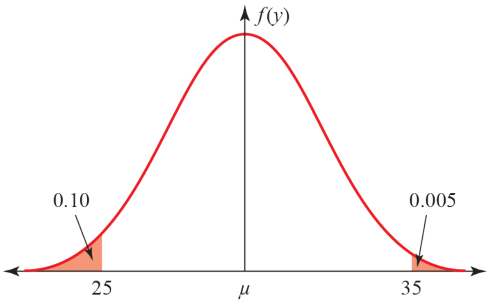
$\mu = 17 - 0.8998 \times 4.329... = 13.104...$

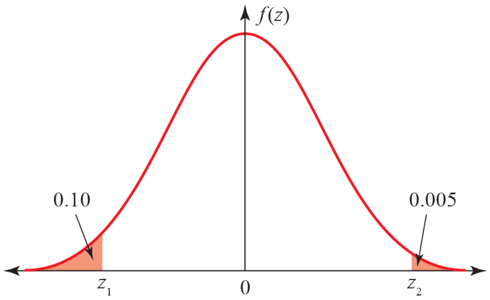
So $\mu = 13.1$ and $\sigma = 4.33$ (3 s.f.)

6 Using the inverse normal function (or the percentage points table),

$$P(Y < 25) = 0.10 \Rightarrow P\left(Z < \frac{25-\mu}{\sigma}\right) = 0.10 \Rightarrow z_1 = -1.28155...$$

$$P(Y > 35) = 0.005 \Rightarrow P\left(Z > \frac{35-\mu}{\sigma}\right) = 0.005 \Rightarrow z_2 = 2.57582...$$





So $-1.2816\sigma = 25 - \mu$ **(1)**

and $2.5758\sigma = 35 - \mu$ **(2)**

(2) – **(1)**: $3.8574\sigma = 10$

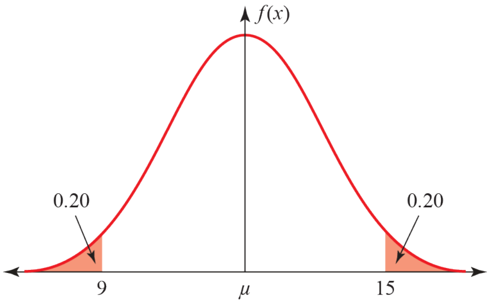
$\sigma = 2.5924...$

Substituting into (**2**):

$\mu = 35 - 2.5758 \times 2.5924... = 28.322...$

So $\mu = 28.3$ and $\sigma = 2.59$ (3 s.f.)

7



By symmetry, $\mu = \frac{1}{2}(9+15) = 12$

$$P(X > 15) = 0.20 \Rightarrow P\left(Z > \frac{15-12}{\sigma}\right) = 0.20$$

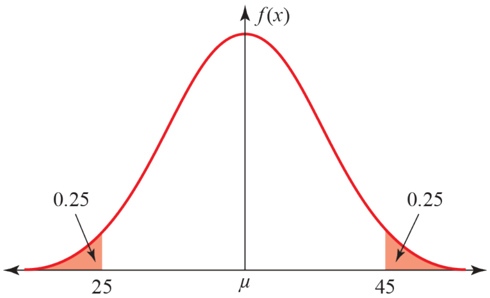
Using the inverse normal function (or the percentage points table), $z = 0.8416...$

so $0.8416 = \dfrac{3}{\sigma}$

$$\sigma = \frac{3}{0.8416} = 3.564...$$

So $\mu = 12$ and $\sigma = 3.56$ (3 s.f.)

8



By symmetry, $\mu = \frac{1}{2}(25+45) = 35$

$$P(X > 45) = 0.25 \Rightarrow P\left(Z > \frac{45-35}{\sigma}\right) = 0.25$$

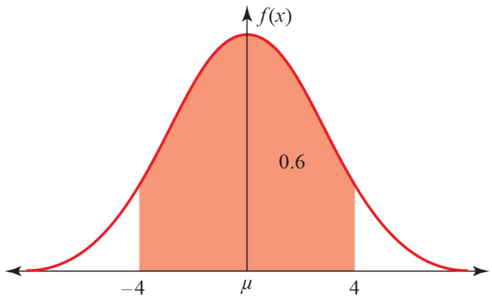
Using the inverse normal function, $z = 0.6744...$

so $0.6744 = \dfrac{10}{\sigma}$

$$\sigma = \frac{10}{0.6744} = 14.82...$$

So $\mu = 35$ and $\sigma = 14.8$ (3 s.f.)

9 $\mu = 0$ (given) so $P(X > 4) = 0.2$ and $P\left(Z > \frac{4-0}{\sigma}\right) = 0.2$



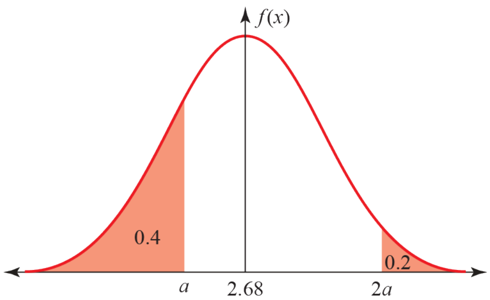
Using the inverse normal function (or the percentage points table), $z = 0.8416...$

so $0.8416 = \frac{4}{\sigma}$

$$\sigma = \frac{4}{0.8416} = 4.752...$$

So $\sigma = 4.75$ (3 s.f.)

10



Using the inverse normal function (or the percentage points table),

$P(X > 2a) = 0.2 \Rightarrow P\left(Z < \frac{2a - 2.68}{\sigma}\right) = 0.2 \Rightarrow z_1 = 0.8416...$

$P(X < a) = 0.4 \Rightarrow P\left(Z < \frac{a - 2.68}{\sigma}\right) = 0.4 \Rightarrow z_2 = -0.2533...$

So $\quad 0.8416\sigma = 2a - 2.68 \quad$ **(1)**

and $\quad -0.2533\sigma = a - 2.68 \quad$ **(2)**

(2) $\times 2$: $\; -0.5066\sigma = 2a - 5.36 \quad$ **(3)**

(1) − **(3)**: $\; 1.3482\sigma = 2.68$

$$\sigma = 1.9878...$$

Substituting into **(2)**:

$a = 2.68 - 0.2533 \times 1.9878... = 2.176...$

So $\sigma = 1.99$ and $a = 2.18$ (3 s.f.)

11 a The distribution is $D \sim N(\mu, 5^2)$.

$P(D > 200) = 0.75 \Rightarrow P(D < 200) = 0.25 \Rightarrow P\left(Z < \frac{200-\mu}{5}\right) = 0.25$

Using the inverse normal function, $z = -0.6744...$

so $-0.6744... = \frac{200-\mu}{5}$

$\mu = 200 + 5 \times 0.6744...$

$= 203.37... = 203 \text{ mm (3 s.f.)}$

b $P(204 < D < 206) = P(D < 206) - P(D < 204)$

$= P\left(Z < \frac{206 - 203.37...}{5}\right) - P\left(Z < \frac{204 - 203.37...}{5}\right)$

$= P(Z < 0.5256) - P(Z < 0.1256)$

$= 0.70041... - 0.54997...$

$= 0.15045 = 0.1504 \text{ (4 d.p.)}$

c $P(D > 205) = P(Z > 0.3256) = 1 - 0.62763... = 0.37237...$
So the probability that all three bowls are greater than 205 mm in diameter is $P(D > 205)^3$, i.e. $(0.37237...)^3 = 0.05163... = 0.0516$ (3 s.f.).

12 a The distribution is $T \sim N(2.5, \sigma)$.

$P(T < 2.55) = 0.65 \Rightarrow P\left(Z < \frac{2.55 - 2.5}{\sigma}\right) = 0.65$

Using the inverse normal function, $z = 0.38532...$

so $\frac{2.55 - 2.5}{\sigma} = 0.38532...$

$0.05 = \sigma \times 0.38532...$

$\sigma = 0.12976... = 0.1298 \text{ (4 d.p.)}$

b $P(2.4 < T < 2.6) = P(T < 2.6) - P(T < 2.4)$ (4 s.f.).

$= P\left(Z < \frac{2.6 - 2.5}{0.12976...}\right) - P\left(Z < \frac{2.4 - 2.5}{0.12976...}\right)$

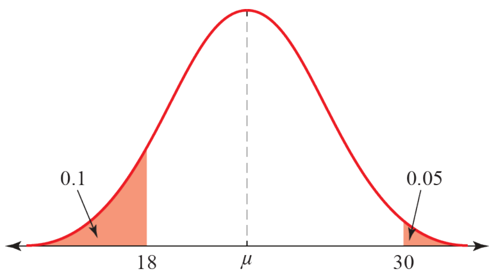
$= P(Z < 0.77065...) - P(Z < -0.77065...)$

$= 0.77954... - 0.22045...$

$= 0.55908... = 0.5591 \text{ (4 d.p.)}$

c Use the binomial distribution $X \sim B(20, 0.5591)$.
Using the binomial CD function,
$P(X \geqslant 15) = 1 - P(X \leqslant 14) = 1 - 93501... = 0.06497...$
So the probability that at least 15 table cloths can be sold is 0.0650 (4 d.p.)

13 a



b $\text{P}(M < 18) = 0.10 \Rightarrow \text{P}\left(Z < \frac{18-\mu}{\sigma}\right) = 0.10 \Rightarrow z_1 = -1.28155...$

$\text{P}(M > 30) = 0.05 \Rightarrow \text{P}\left(Z > \frac{30-\mu}{\sigma}\right) = 0.05 \Rightarrow z_2 = 1.64485...$

So $\quad -1.28155... \times \sigma = 18 - \mu \quad$ **(1)**

and $\quad 1.64485... \times \sigma = 30 - \mu \quad$ **(2)**

(2) − **(1)**: $\quad 2.92640... \times \sigma = 12$

$\sigma = 4.10059...$

Substituting into **(2)**:

$\mu = 30 - 1.64485... \times 4.10059... = 23.25512...$

So $\mu = 23.26$ and $\sigma = 4.101$ (4 s.f.)

c $\text{P}(M > 25) = 1 - \text{P}\left(Z > \frac{25 - 23.25512...}{4.10059...}\right) = 0.33522...$

Use the binomial distribution $X \sim \text{B}(10, 0.33522...)$

Using the binomial CD function,

$\text{P}(X \geqslant 4) = 1 - \text{P}(X \leqslant 3) = 1 - 0.55408... = 0.44591...$

So the probability that at least 4 have a mass greater than 25 kg is 0.4459 (4 d.p.)

14 a $\text{P}(L < 16) = 0.20 \Rightarrow \text{P}\left(Z < \frac{16-\mu}{\sigma}\right) = 0.20 \Rightarrow z_1 = -0.84162...$

$\text{P}(L > 18) = 0.10 \Rightarrow \text{P}\left(Z > \frac{18-\mu}{\sigma}\right) = 0.10 \Rightarrow z_2 = 1.28155...$

So $\quad -0.84162... \times \sigma = 16 - \mu \quad$ **(1)**

and $\quad 1.28155... \times \sigma = 18 - \mu \quad$ **(2)**

(2) − **(1)**: $\quad 2.12317... \times \sigma = 2$

$\sigma = 0.94198...$

Substituting into **(2)**:

$\mu = 18 - 1.28155... \times 0.94198... = 16.79279...$

So $\mu = 16.79$ and $\sigma = 0.9420$ (4 s.f.)

b $\text{P}(L < Q_1) = 0.25 \Rightarrow z_1 = -0.67448... \Rightarrow \frac{Q_1 - 16.79279...}{0.94198...} = -0.67448... \Rightarrow Q_1 = 16.15743...$

$\text{P}(L < Q_3) = 0.75 \Rightarrow z_2 = 0.67448... \Rightarrow \frac{Q_3 - 16.79279...}{0.94198...} = 0.67448... \Rightarrow Q_3 = 17.42814...$

The interquartile range is $Q_3 - Q_1 = 17.42814... - 16.15743... = 1.2701... = 1.27$ (2 d.p.)

Challenge

a Let the quartiles be $Q_3 = \mu + z\sigma$ and $Q_1 = \mu + z\sigma$.
Then the interquartile range is $Q_3 - Q_1 = q = (\mu + z\sigma) - (\mu - z\sigma) = 2z\sigma$
z is such that $\Phi(z) = 0.75$ so, using the inverse normal function, $z = 0.67448...$
So $q = 2 \times 0.67448... \times \sigma = 1.34987... \times \sigma$
and hence $\sigma = 0.74130... \times p = 0.741p$ (3 s.f.)

b Since $q = (\mu + z\sigma) - (\mu - z\sigma) = 2z\sigma$ (i.e. the μ s cancel), q is not dependent on μ.
So it is not possible to write an equation for q in terms of μ, and vice versa.