Conditional probability Mixed exercise 2

1 a $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.35 - 0.2 = 0.55$

b $P(A' \cap B') = 1 - P(A \cup B) = 1 - 0.55 = 0.45$

c $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{0.2}{0.4} = 0.5$

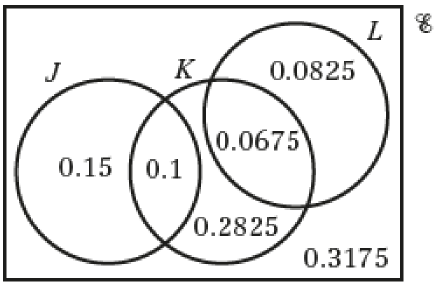
d $P(A'|B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = \dfrac{0.15}{0.35} = 0.429$ (3 s.f.)

2 a Work out each region of the Venn diagram from the information provided in the question.

First, as J and L are mutually exclusive, $P(J \cap L) = \varnothing$
So $P(J \cap K' \cap L') = P(J) - P(J \cap K) = 0.25 - 0.1 = 0.15$

As K and L are independent $P(K \cap L) = P(K) \times P(L) = 0.45 \times 0.15 = 0.0675$
So $P(L \cap K') = P(L) - P(L \cap K) = 0.15 - 0.0625 = 0.0825$
And $P(K \cap J' \cap L') = P(K) - P(J \cap K) - P(K \cap L) = 0.45 - 0.1 - 0.0675 = 0.2825$

Find the outer region by subtracting the sum of all the other regions from 1
$P(J' \cap K' \cap L') = 1 - 0.15 - 0.1 - 0.2825 - 0.0675 - 0.0825 = 0.3175$



b i $P(J \cup K) = 0.15 + 0.1 + 0.2825 + 0.0675 = 0.6$

ii $P(J' \cap L') = 0.2825 + 0.3175 = 0.6$

iii $P(J|K) = \dfrac{P(J \cap K)}{P(K)} = \dfrac{0.1}{0.45} = 0.222$ (3 s.f.)

iv $P(K|J' \cap L') = \dfrac{P(K \cap (J' \cap L'))}{P(J' \cap L')} = \dfrac{0.2825}{0.6} = 0.471$ (3 s.f.)

3 a $P(F \cap S') + P(S \cap F') = P(F) - P(F \cap S) + P(F) - P(F \cap S)$

$$= \frac{35 - 27 + 45 - 27}{60} = \frac{26}{60} = 0.433 \text{ (3 s.f.)}$$

b $P(F|S) = \frac{P(F \cap S)}{P(S)} = \frac{27}{45} = 0.6$

c $P(S|F') = \frac{P(S \cap F')}{P(F')} = \frac{45 - 27}{60 - 35} = \frac{18}{25} = 0.72$

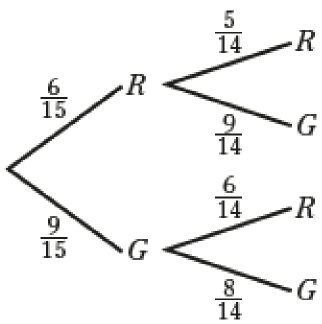
d There are 6 students that study just French and wear glasses ($8 \times 0.75 = 6$) and 9 students that study just Spanish and wear glasses ($18 \times 0.5 = 9$), so the required probability is

$$\text{P(studies one language and wears glasses)} = \frac{6 + 9}{60} = \frac{15}{60} = 0.25$$

e There are 26 students studying one language (from part **a**). Of these, 15 wear glasses (from part **d**).

$$\text{P(wears glasses|studies one language)} = \frac{15}{26} = 0.577 \text{ (3 s.f.)}$$

4 a



b i $P(GG) = \frac{9}{15} \times \frac{8}{14} = \frac{3}{5} \times \frac{4}{7} = \frac{12}{35} = 0.343 \text{ (3 s.f.)}$

ii There are two different ways to obtain balls that are different colours:

$$P(RG) + P(GR) = \left(\frac{6}{15} \times \frac{9}{14}\right) + \left(\frac{9}{15} \times \frac{6}{14}\right) = \frac{2 \times 9}{5 \times 7} = \frac{18}{35} = 0.514 \text{ (3 s.f.)}$$

c There are 4 different outcomes:

$$P(RRR) + P(RGR) + P(GRR) + P(GGR)$$

$$= \left(\frac{6}{15} \times \frac{5}{14} \times \frac{4}{13}\right) + \left(\frac{6}{15} \times \frac{9}{14} \times \frac{5}{13}\right) + \left(\frac{9}{15} \times \frac{6}{14} \times \frac{5}{13}\right) + \left(\frac{9}{15} \times \frac{8}{14} \times \frac{6}{13}\right)$$

$$= \frac{120 + 270 + 270 + 432}{2730} = \frac{1092}{2730} = 0.4$$

4 d The only way for this to occur is to draw a green ball each time. The corresponding probability is:

$$\text{P}(GGGG) = \frac{9}{15} \times \frac{8}{14} \times \frac{7}{13} \times \frac{6}{12} = \frac{3 \times 2}{5 \times 13} = \frac{6}{65} = 0.0923 \text{ (3 s.f.)}$$

5 a Either Colin or Anne must win both sets. Therefore the required probability is:

$$\text{P(match over in two sets)} = (0.7 \times 0.8) + (0.3 \times 0.6) = 0.56 + 0.18 = 0.74$$

b $\text{P(Anne wins}|\text{match over in two sets)} = \frac{0.7 \times 0.8}{0.74} = \frac{0.56}{0.74} = 0.757 \text{ (3 s.f.)}$

c The three ways that Anne can win the match are: wins first set, wins second set; wins first set, loses second set, wins tiebreaker; loses first set, wins second set, wins tiebreaker.

$$\begin{aligned}\text{P(Anne wins match)} &= (0.7 \times 0.8) + (0.7 \times 0.2 \times 0.55) + (0.3 \times 0.4 \times 0.55) \\ &= 0.56 + 0.077 + 0.066 = 0.703\end{aligned}$$

6 a There are 20 kittens with neither black nor white paws (75 – 26 – 14 – 15 = 20).

$$\text{P(neither white or black paws)} = \frac{20}{75} = \frac{4}{15} = 0.267 \text{ (3 s.f.)}$$

b There are 41 kittens with some black paws (26 + 15 = 41).

$$\text{P(black and white paws}|\text{some black paws)} = \frac{15}{41} = 0.366 \text{ (3 s.f.)}$$

c This is selection without replacement (since the first kitten chosen is not put back).

$$\text{P(both kittens have all black paws)} = \frac{26}{75} \times \frac{25}{74} = \frac{13}{3 \times 37} = \frac{13}{111} = 0.117 \text{ (3 s.f.)}$$

d There are 29 kittens with some white paws (14 + 15 = 29).

$$\text{P(both kittens have some white paws)} = \frac{29}{75} \times \frac{28}{74} = \frac{812}{5550} = 0.146 \text{ (3 s.f.)}$$

7 a Using the fact that A and B are independent: $\text{P}(A) \times \text{P}(B) = \text{P}(A \cap B) \Rightarrow \text{P}(B) = \frac{0.12}{0.4} = 0.3$

b Use the addition formula to find $\text{P}(A \cup B)$

$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B) = 0.4 + 0.3 - 0.12 = 0.58$

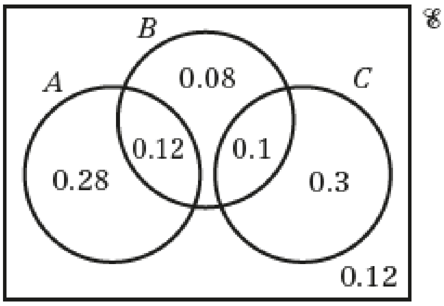
$\text{P}(A' \cap B') = 1 - \text{P}(A \cup B) = 1 - 0.58 = 0.42$

7 c As A and C are mutually exclusive

$P(A \cap B' \cap C') = P(A) - P(A \cap B) = 0.4 - 0.12 = 0.28$
$P(C \cap A' \cap B') = P(C) - P(B \cap C) = 0.4 - 0.1 = 0.3$
$P(B \cap A' \cap C') = P(B) - P(A \cap B) - P(B \cap C) = 0.3 - 0.12 - 0.1 = 0.08$

Find the outer region by subtracting the sum of all the other regions from 1

$P(A' \cap B' \cap C') = 1 - 0.28 - 0.12 - 0.08 - 0.1 - 0.3 = 0.12$



d i $P(B|C) = \dfrac{P(B \cap C)}{P(C)} = \dfrac{0.1}{0.4} = 0.25$

ii The required region must be contained within A, and not include B (the condition on C is irrelevant since A and C are mutually exclusive). Therefore, $P(A \cap (B' \cup C)) = 0.28$

8 a It may be that neither team scores in the match, and it is a 0–0 draw.

b $P(\text{team } A \text{ scores first}) = P(\text{team } A \text{ scores first and wins}) + P(\text{team } A \text{ scores first and does not win})$

So $P(\text{team } A \text{ scores first and does not win}) = 0.6 - 0.48 = 0.12$

c From the question $P(A \text{ wins}|B \text{ scores first}) = 0.3$. Using the multiplication formula gives

$$P(A \text{ wins}|B \text{ scores first}) = \frac{P(A \text{ wins} \cap B \text{ scores first})}{P(B \text{ scores first})} = 0.3$$

$\Rightarrow P(A \text{ wins} \cap B \text{ scores first}) = 0.3 \times 0.35 = 0.105$

Now find the required probability

$$P(B \text{ scores first}|A \text{ wins}) = \frac{P(A \text{ wins} \cap B \text{ scores first})}{P(A \text{ wins})} = \frac{0.105}{0.48 + 0.105} = \frac{0.105}{0.585} = 0.179 \text{ (3 s.f.)}$$

Challenge

a Let $P(A \cap B) = k$

As $P(A \cap B) \leqslant P(B) \Rightarrow k \leqslant 0.2$

A and B could be mutually exclusive, meaning $P(A \cap B) = 0$, so $0 \leqslant k \leqslant 2$

Now, $P(A \cap B') = P(A) - P(A \cap B)$, so $p = 0.6 - k \Rightarrow 0.4 \leqslant p \leqslant 0.6$

Challenge

b Use the fact that $\mathrm{P}(A \cap C) = \mathrm{P}(A \cap B \cap C) + \mathrm{P}(A \cap B' \cap C)$
So $\mathrm{P}(A \cap B' \cap C) = \mathrm{P}(A \cap C) - \mathrm{P}(A \cap B \cap C) = \mathrm{P}(A \cap C) - 0.1$

Consider the range of $\mathrm{P}(A \cap C)$
$\mathrm{P}(A \cap C) \leqslant \mathrm{P}(A) \Rightarrow \mathrm{P}(A \cap C) \leqslant 0.6$

By the multiplication formula $\mathrm{P}(A \cup C) = \mathrm{P}(A) + \mathrm{P}(C) - \mathrm{P}(A \cap C)$
So $\mathrm{P}(A \cap C) = \mathrm{P}(A) + \mathrm{P}(C) - \mathrm{P}(A \cup C) = 1.3 - \mathrm{P}(A \cup C)$
As $\mathrm{P}(A \cup C) \leqslant 1 \Rightarrow \mathrm{P}(A \cap C) \geqslant 0.3$

So $0.3 \leqslant \mathrm{P}(A \cap C) \leqslant 0.6$ and as $\mathrm{P}(A \cap B' \cap C) = \mathrm{P}(A \cap C) - 0.1$ this gives the result that $0.3 - 0.1 \leqslant \mathrm{P}(A \cap B' \cap C) \leqslant 0.6 - 0.1$, so $0.2 \leqslant q \leqslant 0.5$