## Conditional probability 2C

**1 a** The probability $A \cup B$ includes all cases where either event $A$ or event $B$ occurs. So sum the probabilities for these three regions $A \cap B'$, $A \cap B$ and $B \cap A'$.

This gives $P(A \cup B) = 0.3 + 0.12 + 0.28 = 0.7$

**b** The probability that $A$ occurs given that $B$ occurs means that we are only selecting from those situations where $B$ occurs. So the sample space is restricted to just circle $B$. The denominator of the fraction is 0.12 + 0.28 = 0.4. The numerator is when $A$ also occurs i.e. when both $A$ and $B$ occur, which is the region $A \cap B$.

Therefore $P(A|B) = \frac{0.12}{0.4} = 0.3$

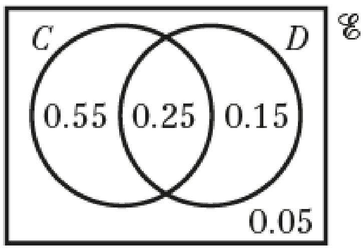
**c** The sample space is restricted to those instances where $A$ has not occurred i.e. the regions $B \cap A'$ or $B' \cap A'$. This means the denominator will be 0.28 + 0.3 = 0.58. The numerator will consist of the cases where $B$ has occurred i.e. $B \cap A'$.

Therefore $P(B|A') = \frac{0.28}{0.58} = 0.483$ (3 s.f.)

**d** The sample space is restricted to those instances where $A$ or $B$ has occurred i.e. the region $A \cup B$. From part **a** this has probability 0.7. The numerator will consist of the cases where B has occurred i.e. either $B \cap A'$ or $B \cap A$.

Therefore $P(B|A \cup B) = \frac{0.28 + 0.12}{0.7} = \frac{0.4}{0.7} = 0.571$ (3 s.f.)

**2 a** Fill in $P(C \cap D) = 0.25$ on the Venn diagram, and then calculate $P(C \cap D') = 0.8 - 0.25 = 0.55$, $P(D \cap C') = 0.4 - 0.25 = 0.15$ and $P(C \cup D)' = 1 - 0.25 - 0.55 - 0.15 = 0.05$



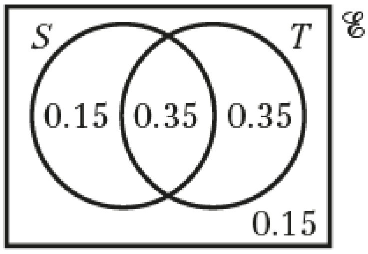

**b i** $P(C \cup D) = 0.55 + 0.25 + 0.15 = 0.95$

**ii** $P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{0.25}{0.4} = 0.625$

**iii** $P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{0.25}{0.8} = 0.3125$

**iv** $P(D'|C') = \frac{P(D' \cap C')}{P(C')} = \frac{0.05}{0.15 + 0.05} = 0.25$

**3 a** Since $S$ and $T$ are independent, $\text{P}(S \cap T) = \text{P}(S) \times \text{P}(T) = 0.5 \times 0.7 = 0.35$, and use this result to fill in the Venn diagram.



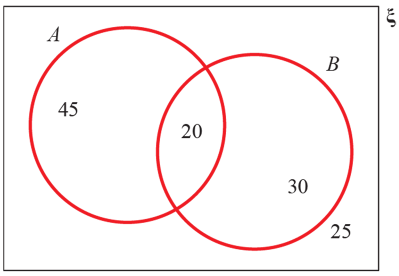

**b i** This is calculated to complete the Venn diagram in part **a**, $\text{P}(S \cap T) = 0.35$

**ii** $\text{P}(S|T) = \dfrac{\text{P}(S \cap T)}{\text{P}(T)} = \dfrac{0.35}{0.7} = 0.5$

**iii** $\text{P}(T|S') = \dfrac{\text{P}(T \cap S')}{\text{P}(S')} = \dfrac{0.35}{0.5} = 0.7$

**iv** $\text{P}(S|(S' \cup T')) = \dfrac{\text{P}(S \cap (S' \cup T'))}{\text{P}(S' \cup T')} = \dfrac{\text{P}(S \cap T')}{0.15 + 0.35 + 0.15} = \dfrac{0.15}{0.65} = 0.231$ (3 s.f.)

**4 a** First produce a Venn diagram with the numbers of people in each region.




The Venn diagram can now be used to find the required probabilities. From the diagram,
$\text{P}(A \cap B') = \dfrac{45}{120} = \dfrac{3}{8} = 0.375$

**b** $\text{P}(A|B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)} = \dfrac{\frac{20}{120}}{\frac{50}{120}} = \dfrac{20}{50} = \dfrac{2}{5} = 0.4$

**c** $\text{P}(B|A') = \dfrac{\text{P}(B \cap A')}{\text{P}(A')} = \dfrac{30}{55} = \dfrac{6}{11} = 0.545$ (3 s.f.)

**d** $\text{P}(A|(A \cup B)) = \dfrac{\text{P}(A \cap (A \cup B))}{\text{P}(A \cup B)} = \dfrac{\text{P}(A)}{\text{P}(A \cup B)} = \dfrac{65}{95} = \dfrac{13}{19} = 0.684$ (3 s.f.)

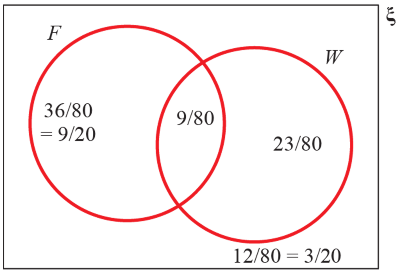
**5 a** Note that 12 cats like neither brand of food. So 80 – 12 = 68 cats like Feskers or Whilix or both. Use this and the other information in the question to calculate $\mathrm{P}(F \cap W)$ as follows:

$$\mathrm{P}(F \cup W) = \mathrm{P}(F) + \mathrm{P}(W) - \mathrm{P}(F \cap W)$$

$$\Rightarrow \mathrm{P}(F \cap W) = \mathrm{P}(F) + \mathrm{P}(W) - \mathrm{P}(F \cup W)$$

$$\text{So } \mathrm{P}(F \cap W) = \frac{45}{80} + \frac{32}{80} - \frac{68}{80} = \frac{9}{80}$$

This is a Venn diagram showing the result:




**b** $\mathrm{P}(F|W) = \dfrac{\mathrm{P}(F \cap W)}{\mathrm{P}(W)} = \dfrac{\frac{9}{80}}{\frac{32}{80}} = \dfrac{9}{32} = 0.281$ (3 s.f.)

**c** $\mathrm{P}(W|F) = \dfrac{\mathrm{P}(F \cap W)}{\mathrm{P}(F)} = \dfrac{\frac{9}{80}}{\frac{45}{80}} = \dfrac{9}{45} = \dfrac{1}{5} = 0.2$

**d** $\mathrm{P}(W'|F') = \dfrac{\mathrm{P}(F' \cap W')}{\mathrm{P}(F')} = \dfrac{\frac{12}{80}}{\frac{23+12}{80}} = \dfrac{12}{35} = 0.343$ (3 s.f.)

**6 a** $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \dfrac{0.2+0.1}{0.2+0.1+0.08+0.12} = \dfrac{0.3}{0.5} = 0.6$

**b** $\mathrm{P}(C|A') = \dfrac{\mathrm{P}(C \cap A')}{\mathrm{P}(A')} = \dfrac{0.1+0.08}{0.1+0.08+0.12+0.15} = \dfrac{0.18}{0.45} = 0.4$

**c** $\mathrm{P}((A \cap B)|C') = \dfrac{\mathrm{P}(A \cap B \cap C')}{\mathrm{P}(C')} = \dfrac{0.2}{0.2+0.2+0.12+0.15} = \dfrac{0.2}{0.67} = 0.299$ (3 s.f.)

**d** $\mathrm{P}(C|(A' \cup B')) = \dfrac{\mathrm{P}(C \cap (A' \cup B'))}{\mathrm{P}(A' \cup B')} = \dfrac{0.05+0.08+0.1}{0.2+0.05+0.08+0.12+0.1+0.15} = \dfrac{0.23}{0.7} = 0.329$ (3 s.f.)

**7 a** The fact that the student must watch at least one of the TV programmes means that the student is selected from a region contained in $A \cup B \cup C$. Therefore this question should be interpreted as:

$$\mathrm{P}(C|(A \cup B \cup C)) = \frac{\mathrm{P}(C \cap (A \cup B \cup C))}{\mathrm{P}(A \cup B \cup C)} = \frac{\mathrm{P}(C)}{\left(\frac{23}{29}\right)} = \frac{\left(\frac{9}{29}\right)}{\left(\frac{23}{29}\right)} = \frac{9}{23} = 0.391 \text{ (3 s.f.)}$$

The other way to do this is to note that only 23 students watch at least one of the TV programmes, and of these 9 watch programme $C$.

**b** The standard method is as follows:

$$\mathrm{P}(\text{exactly two programmes}|A \cup B \cup C) = \frac{\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) + \mathrm{P}(B \cap C) - 3\mathrm{P}(A \cap B \cap C)}{\mathrm{P}(A \cup B \cup C)}$$

$$= \frac{\left(\frac{2}{29} + \frac{0}{29} + \frac{1}{29} - 3\frac{0}{29}\right)}{\left(\frac{23}{29}\right)} = \frac{3}{23} = 0.130 \text{ (3 s.f.)}$$

An alternative method is to note that 2 + 1 = 3 students watch exactly two of the programmes (they watch $A$ and $B$, and $B$ and $C$, respectively) and so 3 out of the 23 students that watch at least one of the TV programmes watch exactly two of the programmes.

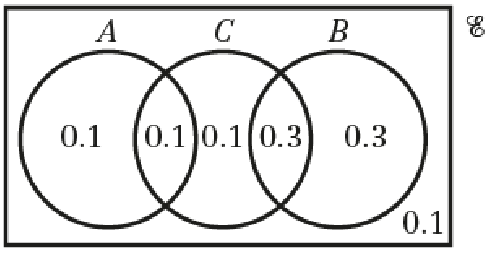
**c** $$\mathrm{P}(B) = \frac{2+7+1}{29} = \frac{10}{29} = 0.345 \text{ (3 s.f.)}$$

$$\mathrm{P}(B|C) = \frac{\mathrm{P}(B \cap C)}{\mathrm{P}(C)} = \frac{\frac{1}{29}}{\frac{9}{29}} = \frac{1}{9} = 0.111 \text{ (3 s.f.)}$$

So $\mathrm{P}(B) \neq \mathrm{P}(B \mid C)$ and the events are not independent.

**8 a** $P(A \cap B) = 0$ since $A$ and $B$ are mutually exclusive.
$P(B \cap C) = P(B) \times P(C) = 0.6 \times 0.5 = 0.3$ since $B$ and $C$ are independent.
$P(B \cap C') = P(B) - P(B \cap C) = 0.6 - 0.3 = 0.3$
As $P(A \cap B) = 0$, $P(A \cup C) = 1 - P(B \cap C') - P(A' \cup B' \cup C') = 1 - 0.3 - 0.1 = 0.6$
$P(A \cup C) = P(A) + P(C) - P(A \cap C)$
$\Rightarrow P(A \cap C) = P(A) + P(C) - P(A \cup C) = 0.2 + 0.5 - 0.6 = 0.1$

Now it is straightforward to work out remaining regions for the Venn diagram
$P(A \cap C') = 0.2 - P(A \cap C) = 0.2 - 0.1 = 0.1$
$P(C \cap A' \cap B') = 0.5 - P(A \cap C) - P(B \cap C) = 0.5 - 0.1 - 0.3 = 0.1$

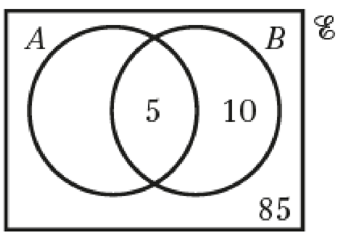



**b i** $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{0.1}{0.5} = 0.2$

**ii** $P(B|C') = \dfrac{P(B \cap C')}{P(C')} = \dfrac{0.3}{0.1 + 0.3 + 0.1} = \dfrac{0.3}{0.5} = 0.6$

**iii** $P(C|(A \cup B)) = \dfrac{P(C \cap (A \cup B))}{P(A \cup B)} = \dfrac{0.1 + 0.3}{0.1 + 0.1 + 0.3 + 0.3} = \dfrac{0.4}{0.8} = 0.5$

**9 a** All of the people who have the disease test positive, which means that there are no people in $A$ who are not in $A \cap B$. There are also 10 people who test positive but do not have the disease. These people lie in $B$ but do not lie in $A$, i.e. they lie in $B \cap A'$. There are 100 – 10 – 5 = 85 people who do not have the disease and do not test positive, so they lie in $A' \cap B'$. Therefore the Venn diagram should show:




**b** $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.05}{0.15} = \dfrac{1}{3} = 0.333$ (3 s.f.)

**c** The test would allow the doctor to find all of the people who have the disease, but only one third of those who tested positive would actually have the disease. This means that two thirds of the people who were told they had the disease would actually not have it.

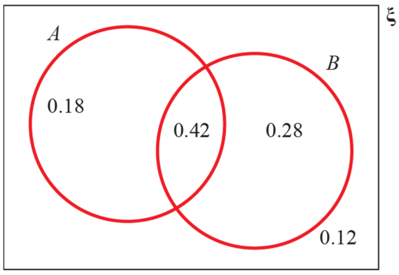
**10 a** Since $\mathrm{P}(A' \cap B') = 0.12$, this means that $\mathrm{P}(A \cup B) = 1 - 0.12 = 0.88$

Now find $\mathrm{P}(A \cap B)$

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\Rightarrow \mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) = 0.6 + 0.7 - 0.88 = 0.42$

This allows a Venn diagram of the probabilties of the two events to be produced, which can be used to answer each part of the question.




$$\mathrm{P}(B|A') = \frac{\mathrm{P}(B \cap A')}{\mathrm{P}(A')} = \frac{0.28}{0.28 + 0.12} = \frac{0.28}{0.4} = 0.7$$

**b** $\mathrm{P}(B|A) = \dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)} = \dfrac{0.42}{0.6} = 0.7$

**c** Since $\mathrm{P}(B|A) = \mathrm{P}(B|A') = \mathrm{P}(B)$, the events $A$ and $B$ are independent.

**11** $\mathrm{P}(A|B) = \mathrm{P}(B')$

$\Rightarrow \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = P(B') = 0.2 + 0.1 = 0.3$

$\Rightarrow \mathrm{P}(A \cap B) = 0.3\mathrm{P}(B)$

$\Rightarrow x = 0.3(x + y)$

The probabilities must sum to 1, so $0.2 + x + y + 0.1 = 1 \Rightarrow x + y = 0.7$

Substituting for $x + y$ gives

$x = 0.3(0.7) = 0.21$

$y = 0.7 - x = 0.7 - 0.21 = 0.49$

**12** $\text{P}(A|B) = \text{P}(A')$

$$\Rightarrow \frac{\text{P}(A \cap B)}{\text{P}(B)} = \text{P}(A')$$

$$\Rightarrow \frac{c}{c+d} = d + 0.2 \qquad \textbf{(1)}$$

The probabilities must sum to 1, so

$$0.3 + c + d + 0.2 = 1 \Rightarrow c + d = 0.5 \qquad \textbf{(2)}$$

Substituting for $c + d$ in the equation (**1**) gives

$$\frac{c}{c+d} = d + 0.2 \Rightarrow c = 0.5d + 0.1 \qquad \textbf{(3)}$$

Substituting this equation for $c$ in equation (**2**) gives

$$0.5d + 0.1 + d = 0.5 \Rightarrow 1.5d = 0.4 \Rightarrow d = \frac{4}{15}$$

Finally, using equation (**3**) gives

$$c = 0.5 \times \frac{4}{15} + 0.1 = \frac{4}{30} + \frac{1}{10} = \frac{4}{30} + \frac{3}{30} = \frac{7}{30}$$