Vectors 12C

1 a i $|\overrightarrow{OA}| = \sqrt{1+4^2+8^2} = \sqrt{81} = 9$

$|\overrightarrow{OB}| = \sqrt{4^2+4^2+7^2} = \sqrt{81} = 9$

$\Rightarrow |\overrightarrow{OA}| = |\overrightarrow{OB}|$

ii $|\overrightarrow{AC}| = |\overrightarrow{OC} - \overrightarrow{OA}| = |9\mathbf{i} + 4\mathbf{j} + 22\mathbf{k}|$

$= \sqrt{9^2+4^2+22^2} = \sqrt{581}$

$|\overrightarrow{BC}| = |\overrightarrow{OC} - \overrightarrow{OB}| = |6\mathbf{i} - 4\mathbf{j} + 23\mathbf{k}|$

$= \sqrt{6^2+4^2+23^2} = \sqrt{581}$

$\Rightarrow |\overrightarrow{AC}| = |\overrightarrow{BC}|$

b The quadrilateral *OACB* has two pairs of equal adjacent sides, so it is a kite.

2 a Let *O* be the fixed origin.

$|\overrightarrow{AB}| = |\overrightarrow{OB} - \overrightarrow{OA}| = |2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}|$

$= \sqrt{2^2+3^2+2^2} = \sqrt{17}$

$|\overrightarrow{AC}| = |\overrightarrow{OC} - \overrightarrow{OA}| = |6\mathbf{j}| = 6$

$|\overrightarrow{BC}| = |\overrightarrow{OC} - \overrightarrow{OB}| = |-2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}|$

$= \sqrt{2^2+3^2+2^2} = \sqrt{17}$

So $|\overrightarrow{AB}| = |\overrightarrow{BC}|$ and the triangle is isosceles.

b If *AC* is the base of the triangle, then the height, *h*, will be given by:

$\left(\frac{1}{2}|\overrightarrow{AC}|\right)^2 + h^2 = \left(|\overrightarrow{AB}|\right)^2$

$9 + h^2 = 17$

$h = \sqrt{8} = 2\sqrt{2}$

Area of triangle *ABC*

$= \frac{1}{2} \times 6 \times 2\sqrt{2} = 6\sqrt{2}$

c For *ABCD* to be a parallelogram, there are three possibilities:

i $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are parallel and equal in magnitude.

Hence $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$

$\overrightarrow{OD} = (2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + (-2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$

$= 4\mathbf{j} + 7\mathbf{k}$

Coordinates of *D* are (0, 4, 7).

ii $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are parallel and equal in magnitude.

Hence $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{AB}$

$\overrightarrow{OD} = (2\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) + (2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

$= 4\mathbf{i} + 10\mathbf{j} + 3\mathbf{k}$

Coordinates of *D* are (4, 10, 3).

iii $\overrightarrow{AD}$ and $\overrightarrow{CB}$ are parallel and equal in magnitude.

Hence $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{CB}$

$\overrightarrow{OD} = (2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$

$= 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

Coordinates of *D* are (4, −2, 3).

3 a Let *O* be the fixed origin.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (11\mathbf{i} + 2\mathbf{j} - 9\mathbf{k}) - (7\mathbf{i} + 12\mathbf{j} - \mathbf{k})$

$= 4\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$

$= 2(2\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= (8\mathbf{i} + \mathbf{j} + 15\mathbf{k}) - (14\mathbf{i} - 14\mathbf{j} + 3\mathbf{k})$

$= -6\mathbf{i} + 15\mathbf{j} + 12\mathbf{k}$

$= -3(2\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$

$\overrightarrow{CD} = -\frac{3}{2}\overrightarrow{AB}$, so *AB* is parallel to *CD*.

$AB : CD = 2 : 3$

3 b $\overrightarrow{BC} = 3\mathbf{i} - 16\mathbf{j} + 12\mathbf{k}$
$\overrightarrow{AD} = \mathbf{i} - 11\mathbf{j} + 16\mathbf{k}$

BC is not parallel to *AD*. So *ABCD* is a quadrilateral with one pair of parallel sides. So it is a trapezium.

4 $(3a+b)\mathbf{i} + \mathbf{j} + ac\mathbf{k} = 7\mathbf{i} - b\mathbf{j} + 4\mathbf{k}$

Comparing coefficients of **j**:
$b = -1$

Comparing coefficients of **i**:
$3a + b = 7 \Rightarrow 3a - 1 = 7$

$a = \frac{8}{3}$

Comparing coefficients of **k**:

$ac = 4 \Rightarrow \frac{8}{3}c = 4$

$c = \frac{3}{2}$

5 $\triangle OAB$ is isosceles.

If $|\overrightarrow{OA}| = |\overrightarrow{OB}|$:

$\sqrt{10^2 + 23^2 + 10^2} = \sqrt{p^2 + 14^2 + 22^2}$

$729 = p^2 + 680$

$p^2 = 49$

$p = \pm 7$

If $|\overrightarrow{OB}| = |\overrightarrow{AB}|$:

$\overrightarrow{AB} = (p-10)\mathbf{i} + 37\mathbf{j} - 32\mathbf{k}$

$\sqrt{p^2 + 14^2 + 22^2} = \sqrt{(p-10)^2 + 37^2 + 32^2}$

$p^2 + 680 = (p-10)^2 + 1369 + 1024$

$p^2 - (p-10)^2 = 2393 - 680$

$p^2 - (p^2 - 20p + 100) = 1713$

$20p = 1813$

$p = \frac{1813}{20}$

If $|\overrightarrow{OA}| = |\overrightarrow{AB}|$:

$\sqrt{729} = \sqrt{(p-10)^2 + 37^2 + 32^2}$

$729 = (p-10)^2 + 1369 + 1024$

$0 = (p-10)^2 + 2393 - 729$

$0 = p^2 - 20p + 100 + 1664$

$0 = p^2 - 20p + 1764$

$b^2 - 4ac < 0$

So there are no solutions for p if $|\overrightarrow{OA}| = |\overrightarrow{AB}|$.

The three possible positions for *B* are (7, 14, −22), (−7, 14, −22) and $\left(\frac{1813}{20}, 14, -22\right)$.

6 **a** $|\overrightarrow{AB}| = \sqrt{7^2 + 1 + 2^2} = \sqrt{54}$

$|\overrightarrow{BC}| = \sqrt{1 + 5^2} = \sqrt{26}$

$|\overrightarrow{AC}| = |\overrightarrow{AB} + \overrightarrow{BC}| = \sqrt{6^2 + 1 + 7^2} = \sqrt{86}$

$$\cos \angle ABC = \frac{54 + 26 - 86}{2 \times \sqrt{54} \times \sqrt{26}} = -0.080...$$

$\angle ABC = 94.59...°$

Area of triangle

$= \frac{1}{2} \times \sqrt{54} \times \sqrt{26} \times \sin 94.59...°$

$= 18.67$ (2 d.p.)

b Triangles ABC and ADE are similar with a side ratio of 1 : 3.

So area of triangle ADE

$= 9 \times$ area of triangle ABC

$= 168.07$ (2 d.p.)

7 Suppose there is a point of intersection, H, of OF and AG.

$\overrightarrow{OH} = r\overrightarrow{OF}$ for some scalar r.

$\overrightarrow{AH} = s\overrightarrow{AG}$ for some scalar s.

But $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OA} + s\overrightarrow{AG}$

so $r\overrightarrow{OF} = \overrightarrow{OA} + s\overrightarrow{AG}$ (1)

Now $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BD} + \overrightarrow{DF} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

and $\overrightarrow{AG} = \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{BG} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$

So (1) becomes

$r(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a} + s(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

Comparing coefficients of **a**:

$r = 1 - s$

Comparing coefficients of **b**:

$r = s$

So $r = s = \frac{1}{2}$

$\overrightarrow{OH} = \frac{1}{2}\overrightarrow{OF}$ and $\overrightarrow{AH} = \frac{1}{2}\overrightarrow{AG}$

So H is the midpoint of OF and of AG, and the diagonals bisect each other.

8 $\overrightarrow{FP} = \overrightarrow{FB} + \overrightarrow{BO} + \overrightarrow{OA} + \overrightarrow{AP}$

$= -\mathbf{c} - \mathbf{b} + \mathbf{a} + \frac{4}{3}\overrightarrow{AM}$

But $\overrightarrow{AM} = \overrightarrow{AO} + \frac{3}{4}\overrightarrow{OE}$

$= -\mathbf{a} + \frac{3}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

$= -\frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b} + \frac{3}{4}\mathbf{c}$

So $\overrightarrow{FP} = -\mathbf{c} - \mathbf{b} + \mathbf{a} + \frac{4}{3}\left(-\frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b} + \frac{3}{4}\mathbf{c}\right)$

$= \frac{2}{3}\mathbf{a}$

$\overrightarrow{PE} = \overrightarrow{PA} + \overrightarrow{AG} + \overrightarrow{GE}$

$= -\frac{4}{3}\overrightarrow{AM} + \mathbf{c} + \mathbf{b}$

$= -\frac{4}{3}\left(\overrightarrow{AO} + \frac{3}{4}\overrightarrow{OE}\right) + \mathbf{c} + \mathbf{b}$

$= \frac{4}{3}\mathbf{a} - \mathbf{a} = \frac{1}{3}\mathbf{a}$

Therefore FP and PE are parallel, so P lies on FE.

$FP : PE = \frac{2}{3}|\mathbf{a}| : \frac{1}{3}|\mathbf{a}| = 2 : 1$

Challenge

1 $p\mathbf{a}+q\mathbf{b}+r\mathbf{c}=\begin{pmatrix} p+2q-5r \\ 3r \\ 4p-3q+r \end{pmatrix}=\begin{pmatrix} 28 \\ -12 \\ -4 \end{pmatrix}$

Comparing coefficients of **b**:
$r=-4$

Comparing coefficients of **a**:
$p+2q+20=28 \Rightarrow p+2q=8 \quad (1)$

Comparing coefficients of **c**:
$4p-3q-4=-4 \Rightarrow 4p-3q=0 \quad (2)$

Substituting for p in (2):
$4(8-2q)-3q=0 \Rightarrow q=\frac{32}{11}$

Substituting for q in (1):
$p+\frac{64}{11}=8 \Rightarrow p=\frac{24}{11}$

2 $\overrightarrow{OM}=\frac{1}{2}\mathbf{a}+\mathbf{b}+\mathbf{c}$

$\overrightarrow{BN}=\mathbf{a}-\mathbf{b}+\frac{1}{2}\mathbf{c}$

$\overrightarrow{AF}=-\mathbf{a}+\mathbf{b}+\mathbf{c}$

Suppose there is a point of intersection, X, of OM and AF.

$\overrightarrow{AX}=r\overrightarrow{AF}=r(-\mathbf{a}+\mathbf{b}+\mathbf{c})$ for scalar r.

$\overrightarrow{OX}=s\overrightarrow{OM}=s\left(\frac{1}{2}\mathbf{a}+\mathbf{b}+\mathbf{c}\right)$ for scalar s.

But $\overrightarrow{OX}=\overrightarrow{OA}+\overrightarrow{AX}=\mathbf{a}+r(-\mathbf{a}+\mathbf{b}+\mathbf{c})$

so $s\left(\frac{1}{2}\mathbf{a}+\mathbf{b}+\mathbf{c}\right)=\mathbf{a}+r(-\mathbf{a}+\mathbf{b}+\mathbf{c})$

Comparing coefficients of **a** and **b**:
$\frac{1}{2}s=1-r$ and $s=r$

So $r=s=\frac{2}{3}$

Suppose there is a point of intersection, Y, of BN and AF**.**

$\overrightarrow{AY}=p\overrightarrow{AF}=p(-\mathbf{a}+\mathbf{b}+\mathbf{c})$ for scalar p.

$\overrightarrow{BY}=q\overrightarrow{BN}=q\left(\mathbf{a}-\mathbf{b}+\frac{1}{2}\mathbf{c}\right)$ for scalar q.

But $\overrightarrow{BY}=\overrightarrow{BA}+\overrightarrow{AY}=\mathbf{a}-\mathbf{b}+p(-\mathbf{a}+\mathbf{b}+\mathbf{c})$

so $q\left(\mathbf{a}-\mathbf{b}+\frac{1}{2}\mathbf{c}\right)=\mathbf{a}-\mathbf{b}+p(-\mathbf{a}+\mathbf{b}+\mathbf{c})$

Comparing coefficients of **a** and **c**:
$q=1-p$ and $q=2p$

So $p=\frac{1}{3}, q=\frac{2}{3}$

$\overrightarrow{AX}=\frac{2}{3}\overrightarrow{AF}$ and $\overrightarrow{AY}=\frac{1}{3}\overrightarrow{AF}$

So the line segments OM and BN trisect the diagonal AF.