Integration 11H

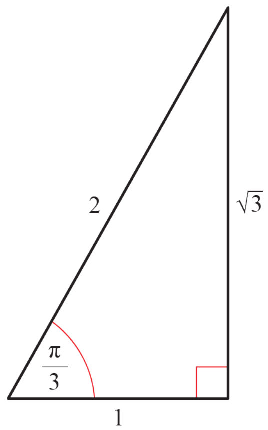
1 a Area $= \int_0^1 \frac{2}{1+x}\,dx = \left[2\ln|1+x|\right]_0^1$

$= (2\ln 2) - (2\ln 1)$

$\therefore$ Area $= 2\ln 2$

b Area $= \int_0^{\frac{\pi}{3}} \sec x\,dx$

$= \left[\ln|\sec x + \tan x|\right]_0^{\frac{\pi}{3}}$



$= [\ln(2+\sqrt{3})] - [\ln(1)]$

$\therefore$ Area $= \ln(2+\sqrt{3})$

c Area $= \int_1^2 \ln x\,dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$\therefore \text{Area} = [x\ln x]_1^2 - \int_1^2 x \times \frac{1}{x}\,dx$

$= (2\ln 2) - (0) - [x]_1^2$

$= 2\ln 2 - 1$

d Area $= \int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$

$= [\sec x]_0^{\frac{\pi}{4}}$

$= (\sqrt{2}) - (1)$

$\therefore \text{Area} = \sqrt{2} - 1$

e Area $= \int_0^2 x\sqrt{4-x^2}\,dx$

Let $y = (4-x^2)^{\frac{3}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{3}{2}(4-x^2)^{\frac{1}{2}} \times (-2x) = -3x(4-x^2)^{\frac{1}{2}}$

$\therefore$ Area $= \left[-\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right]_0^2$

$= (0) - \left(-\frac{1}{3} \times 2^3\right) = \frac{8}{3}$

2 a $f(x) = \frac{4x-1}{(x+2)(2x+1)}$

$\frac{4x-1}{(x+2)(2x+1)} = \frac{A}{x+2} + \frac{B}{2x+1}$

$4x-1 = A(2x+1) + B(x+2)$

$x = -2 \Rightarrow -9 = -3A \Rightarrow A = 3$

$x = -\frac{1}{2} \Rightarrow -3 = \frac{3}{2}B \Rightarrow B = -2$

$f(x) = \frac{3}{x+2} - \frac{2}{2x+1}$

$\int_0^2 \left(\frac{3}{x+2} - \frac{2}{2x+1}\right) dx$

$= \left[3\ln(x+2) - \ln(2x+1)\right]_0^2$

$= 3\ln 4 - \ln 5 - 3\ln 2 + \ln 1$

$= \ln 64 - \ln 5 - \ln 8$

$= \ln\frac{8}{5}$

2 b $\dfrac{x}{(x+1)^2} \equiv \dfrac{A}{(x+1)^2} + \dfrac{B}{x+1}$

$\Rightarrow x \equiv A + B(x+1)$

Compare coefficient of x: $1 = B \Rightarrow B = 1$

Compare constants: $0 = A + B \Rightarrow A = -1$

$\therefore$ area $= \int_0^2 \dfrac{x}{(x+1)^2}\,dx$

$= \int_0^2 \left(\dfrac{1}{x+1} - \dfrac{1}{(x+1)^2} \right) dx$

$= \left[\ln|x+1| + \dfrac{1}{x+1} \right]_0^2$

$= \left(\ln 3 + \dfrac{1}{3} \right) - (\ln 1 + 1)$

$= \ln 3 - \dfrac{2}{3}$

c Area $= \int_0^{\frac{\pi}{2}} x \sin x \,dx$

$u = x \Rightarrow \dfrac{du}{dx} = 1$

$\dfrac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\therefore$ area $= [-x\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x)\,dx$

$= \left(-\dfrac{\pi}{2}\cos\dfrac{\pi}{2} \right) - (0) + \int_0^{\frac{\pi}{2}} \cos x \,dx$

$= 0 + [\sin x]_0^{\frac{\pi}{2}}$

$= \left(\sin\dfrac{\pi}{2} - 0 \right)$

$= 1$

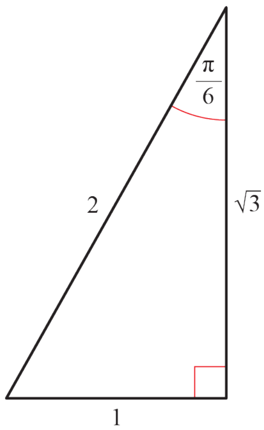
d Area $= \int_0^{\frac{\pi}{6}} \cos x \sqrt{2\sin x + 1}\,dx$

Let $y = (2\sin x + 1)^{\frac{3}{2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3}{2}(2\sin x + 1)^{\frac{1}{2}} \times 2\cos x$

$= 3\cos x (2\sin x + 1)^{\frac{1}{2}}$

$\therefore$ area $= \left[\dfrac{1}{3}(2\sin x + 1)^{\frac{3}{2}} \right]_0^{\frac{\pi}{6}}$



$= \left(\dfrac{1}{3} 2^{\frac{3}{2}} \right) - \left(\dfrac{1}{3} 1^{\frac{3}{2}} \right)$

$= \dfrac{2\sqrt{2}}{3} - \dfrac{1}{3}$

$= \dfrac{2\sqrt{2} - 1}{3}$

2 **e** Area $= \int_0^{\ln 2} x\mathrm{e}^{-x}\,\mathrm{d}x$

$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-x} \Rightarrow v = \mathrm{e}^{-x}$

$\therefore$ area $= \left[-x\mathrm{e}^{-x}\right]_0^{\ln 2} - \int_0^{\ln 2}(-\mathrm{e}^{-x})\,\mathrm{d}x$

$= (-\ln 2 \times \mathrm{e}^{-\ln 2}) - (0) + \int_0^{\ln 2} \mathrm{e}^{-x}\,\mathrm{d}x$

$= -\ln 2 \times \frac{1}{2} + \left[-\mathrm{e}^{-x}\right]_0^{\ln 2}$

$= -\frac{1}{2}\ln 2 + (-\mathrm{e}^{-\ln 2}) - (-\mathrm{e}^{-0})$

$= -\frac{1}{2}\ln 2 - \frac{1}{2} + 1$

$= \frac{1}{2}(1 - \ln 2)$

3 Area $= \int_1^2 \frac{4x+3}{(x+2)(2x-1)}\,\mathrm{d}x$

$\frac{4x+3}{(x+2)(2x-1)} = \frac{A}{x+2} + \frac{B}{2x-1}$

$4x+3 = A(2x-1) + B(x+2)$

Let $x = -2 : -5 = -5A \Rightarrow A = 1$

Let $x = \frac{1}{2} : 5 = \frac{5}{2}B \Rightarrow B = 2$

Area $= \int_1^2 \left(\frac{1}{x+2} + \frac{2}{2x-1}\right)\mathrm{d}x$

$= \left[\ln|x+2| + \ln|2x-1|\right]_1^2$

$= (\ln 4 + \ln 3) - (\ln 3 + \ln 1)$

$= \ln 4$

4 Area $= \int_2^4 \left(\mathrm{e}^{0.5x} + \frac{1}{x}\right)\mathrm{d}x$

$= \left[2\mathrm{e}^{0.5x} + \ln|x|\right]_2^4$

$= (2\mathrm{e}^2 + \ln 4) - (2\mathrm{e} + \ln 2)$

$= 2\mathrm{e}^2 - 2\mathrm{e} + \ln\frac{4}{2}$

$= 2\mathrm{e}^2 - 2\mathrm{e} + \ln 2$

5 a $\mathrm{g}(x) = 0 \Rightarrow A(0,0), B(\pi,0), C(2\pi,0)$

b $I = \int x\sin x\,\mathrm{d}x$

Let $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$

$I = -x\cos x + \int \cos x\,\mathrm{d}x$

$= -x\cos x + \sin x + c$

Area between A and B:

$\left[-x\cos x + \sin x\right]_0^{\pi} = \pi$

Area between B and C:

$\left[-x\cos x + \sin x\right]_{\pi}^{2\pi} = -2\pi - \pi = -3\pi$

Total area $= \pi + 3\pi = 4\pi$

6 a $I = \int x^2 \ln x\,\mathrm{d}x$

Let $u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \Rightarrow v = \frac{x^3}{3}$

$I = \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \times \frac{1}{x}\,\mathrm{d}x$

$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$

b $x^2 \ln x = 0 \Rightarrow x = 0$ or 1

Area between $x = 0$ and $x = 1$:

$\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_0^1 = -\frac{1}{9}$

Area between $x = 1$ and $x = 2$:

$\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^2 = \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) + \frac{1}{9}$

$= \frac{8}{3}\ln 2 - \frac{7}{9}$

Total area $= \frac{8}{3}\ln 2 - \frac{7}{9} + \frac{1}{9} = \frac{8}{3}\ln 2 - \frac{2}{3}$

$= \frac{2}{3}(4\ln 2 - 1)$

7 a $y = 3\cos x\sqrt{\sin x + 1}$

Curve crosses the x axis when $y = 0$.

$\cos x = 0$ or $\sin x = -1$

$x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

$A\left(-\frac{\pi}{2}, 0\right), B\left(\frac{\pi}{2}, 0\right), C\left(\frac{3\pi}{2}, 0\right)$

Curve crosses the y axis when $x = 0$.

So $D(0, 3)$.

b $I = \int 3\cos x\sqrt{\sin x + 1}\ \mathrm{d}x$

Let $u = \sin x + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$

$I = \int 3\sqrt{u}\ \mathrm{d}u = 2u^{\frac{3}{2}} + c$

$= 2(\sin x + 1)^{\frac{3}{2}} + c$

c $R_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\cos x\sqrt{\sin x + 1}\ \mathrm{d}x$

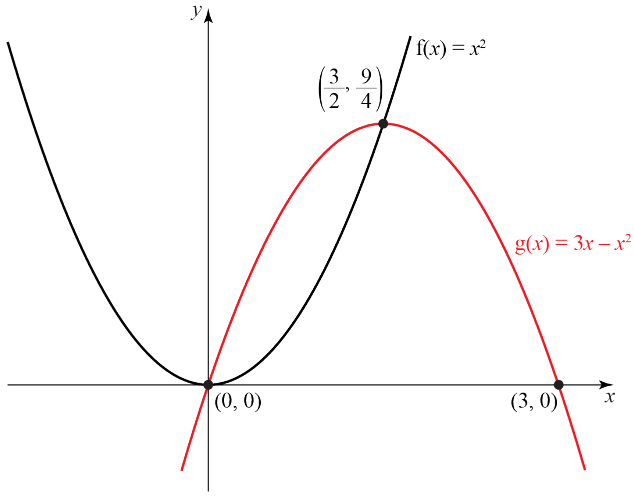
$\left[2(\sin x + 1)^{\frac{3}{2}}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\sqrt{2} = \sqrt{32}$

$R_2 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3\cos x\sqrt{\sin x + 1}\ \mathrm{d}x$

$\left[2(\sin x + 1)^{\frac{3}{2}}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -4\sqrt{2} = -\sqrt{32}$

$R_1 = R_2 = \sqrt{32}$, so $a = 32$.

8 a



$f(x) = g(x) \Rightarrow x^2 = 3x - x^2$

$2x^2 - 3x = 0$

$x(2x - 3) = 0$

$x = 0$ or $\frac{3}{2}$

$x = 0 \Rightarrow y = 0$

$x = \frac{3}{2} \Rightarrow y = \frac{9}{4}$

Points of intersection are

$(0, 0)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$

b Area under $f(x)$ between 0 and $\frac{3}{2}$:

$\int_0^{\frac{3}{2}} x^2\ \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^{\frac{3}{2}} = \frac{27}{24} = \frac{9}{8}$

Area under $g(x)$ between 0 and $\frac{3}{2}$:

$\int_0^{\frac{3}{2}} 3x - x^2\ \mathrm{d}x = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^{\frac{3}{2}}$

$= \frac{27}{8} - \frac{27}{24} = \frac{9}{4}$

Area between the two curves

$= \frac{9}{4} - \frac{9}{8} = \frac{9}{8}$

9 a Points of intersection are when:

$2\cos x + 2 = -2\cos x + 4$

$4\cos x = 2$

$\cos x = \frac{1}{2} \Rightarrow x = -\frac{\pi}{3}, \frac{\pi}{3}$ or $\frac{5\pi}{3}$

$x = -\frac{\pi}{3}, \frac{\pi}{3}$ or $\frac{5\pi}{3} \Rightarrow y = 3$

$A\left(-\frac{\pi}{3}, 3\right), B\left(\frac{\pi}{3}, 3\right), C\left(\frac{5\pi}{3}, 3\right)$

b $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos x + 2)\, dx$

$= [2\sin x + 2x]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

$= \left(\sqrt{3} + \frac{2\pi}{3}\right) - \left(-\sqrt{3} - \frac{2\pi}{3}\right) = 2\sqrt{3} + \frac{4\pi}{3}$

$= 2\sqrt{3} + \frac{4\pi}{3}$

$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (-2\cos x + 4)\, dx$

$= [-2\sin x + 4x]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

$= \left(-\sqrt{3} + \frac{4\pi}{3}\right) - \left(\sqrt{3} - \frac{4\pi}{3}\right)$

$= -2\sqrt{3} + \frac{8\pi}{3}$

$R_1 = 2\sqrt{3} + \frac{4\pi}{3} - \left(-2\sqrt{3} + \frac{8\pi}{3}\right)$

$= 4\sqrt{3} - \frac{4\pi}{3}$

$a = 4, b = -4, c = 3$ (or $a = 4, b = 4, c = -3$)

c $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2\cos x + 2)\, dx$

$= [2\sin x + 2x]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$

$= \left(-\sqrt{3} + \frac{10\pi}{3}\right) - \left(\sqrt{3} + \frac{2\pi}{3}\right)$

$= -2\sqrt{3} + \frac{8\pi}{3}$

$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (-2\cos x + 4)\, dx$

$= [-2\sin x + 4x]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$

$= \left(\sqrt{3} + \frac{20\pi}{3}\right) - \left(-\sqrt{3} + \frac{4\pi}{3}\right)$

$= 2\sqrt{3} + \frac{16\pi}{3}$

$R_2 = \left(2\sqrt{3} + \frac{16\pi}{3}\right) - \left(-2\sqrt{3} + \frac{8\pi}{3}\right)$

$= 4\sqrt{3} + \frac{8\pi}{3}$

$R_2 : R_1 = 4\sqrt{3} + \frac{8\pi}{3} : 4\sqrt{3} - \frac{4\pi}{3}$

$= \left(3\sqrt{3} + 2\pi\right) : \left(3\sqrt{3} - \pi\right)$

10 $y = \sin\theta$

Area under curve $= 2\int_0^{\pi} \sin\theta\, d\theta$ because of the symmetry of the curve.

$= 2[-\cos\theta]_0^{\pi} = 2 + 2 = 4$

$y = \sin 2\theta$

Area under curve $= 4\int_0^{\frac{\pi}{2}} \sin 2\theta\, d\theta$ because of the symmetry of the curve.

$= 4\left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}} = 2 + 2 = 4$

11 a At A, $\cos x = \sin x$

$\tan x = 1 \Rightarrow x = \frac{\pi}{4} \Rightarrow y = \frac{1}{\sqrt{2}}$

$A\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

b i $R_1 =$ Area under $y = \cos x$ − Area under $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$

$R_1 = \int_0^{\frac{\pi}{4}} \cos x \, dx - \int_0^{\frac{\pi}{4}} \sin x \, dx$

$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$

$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$

ii $R_2 = 2 \times$ Area under $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$

$R_2 = 2\int_0^{\frac{\pi}{4}} -\cos x \, dx$

$= 2\left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$

iii $R_3 = \int_{\frac{\pi}{4}}^{\pi} \sin x \, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$

$= [-\cos x]_{\frac{\pi}{4}}^{\pi} - [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= 1 + \frac{1}{\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}}\right) = \sqrt{2}$

c $R_1 : R_2 = (\sqrt{2} - 1) : (2 - \sqrt{2})$

$= (\sqrt{2} - 1)(2 + \sqrt{2}) : (2 - \sqrt{2})(2 + \sqrt{2})$

$= \sqrt{2} : 2$

12 Area $= \int y \frac{dx}{dt} = \int_0^{\sqrt[3]{4}} t^2 (3t^2) dt$

$= \frac{3}{5}(\sqrt[3]{4})^5 = \frac{3}{5}\left(2^{\frac{10}{3}}\right)$

$= \frac{3}{5}(2^3)\left(2^{\frac{1}{3}}\right) = \frac{24}{5}\sqrt[3]{2}$

$\Rightarrow k = \frac{24}{5}$

13 Area $= \int y \frac{dx}{dt} = \int_0^{\frac{\pi}{2}} \sin 2t(\cos t) dt$

Using $\sin 2t = \cos t \sin t$:

Area $= \int_0^{\frac{\pi}{2}} \sin t(\cos^2 t) dt$

Let $u = \cos t \Rightarrow \frac{du}{dt} = -\sin t$

$\Rightarrow dt = \frac{1}{-\sin t} du$

Area $= -2\int_0^{\frac{\pi}{2}} u^2 du = \left[-\frac{2u^3}{3}\right]_0^{\frac{\pi}{2}}$

$= \left[-\frac{2\cos^3 t}{3} + c\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{2}{3}\right) - (0)$

$= \frac{2}{3}$

14 a P is at point $t = 2$

$x = (2 + 1)^2 = 9$

$y = \frac{1}{2}(2^3) + 3 = 7$

$(9, 7)$

Equation of normal at P:

$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{3}{2}t^2, \quad \frac{dx}{dt} = 2t + 2$

$\frac{dy}{dx} = \frac{\frac{3}{2}t^2}{2t + 2} = \frac{\frac{3}{2}(2)^2}{4 + 2} = 1$

14 a Gradient of normal is negative reciprocal of derivative at P $\therefore m = -1$

$$y - y_1 = m(x - x_1)$$
$$y - 7 = -1(x - 9)$$
$$y + x = 16$$

b

$$\text{Area} = \int_0^2 \left(\frac{1}{2}t^3 + 3\right)(2t + 2)\,\mathrm{d}t + \int_9^{16} (16 - x)\,\mathrm{d}x$$
$$= \int_0^2 (t^4 + t^3 + 6t + 6)\,\mathrm{d}t + \int_9^{16} (16 - x)\,\mathrm{d}x$$
$$= \left[\frac{t^5}{5} + \frac{t^4}{4} + 3t^2 + 6t\right]_0^2 + \left[16x - \frac{x^2}{2}\right]_9^{16}$$
$$= 34.4 + 24.5$$
$$= 58.9$$

Challenge

Curves intersect at $\sin 2x = \cos x$

$$2\sin x \cos x = \cos x$$
$$\sin x = \frac{1}{2}, \cos x \neq 0$$
$$x = \frac{\pi}{6}$$

Shaded area

$$= \int_0^{\frac{\pi}{6}} (\cos x - \sin 2x)\ \mathrm{d}x + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin 2x - \cos x)\ \mathrm{d}x$$
$$= \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}} + \left[-\frac{1}{2}\cos 2x - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$
$$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(0 + \frac{1}{2}\right) + \left(0 - \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{4} - \frac{1}{2}\right)$$
$$= 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2}}$$
$$= \frac{2 - \sqrt{2}}{2}$$