Integration 11B

1 a $\int \sin(2x+1)\,dx = -\frac{1}{2}\cos(2x+1)+c$

b $\int 3e^{2x}\,dx = \frac{3}{2}e^{2x}+c$

c $\int 4e^{x+5}\,dx = 4e^{x+5}+c$

d $\int \cos(1-2x)\,dx = -\frac{1}{2}\sin(1-2x)+c$

OR Let $y=\sin(1-2x)$

then $\frac{dy}{dx}=\cos(1-2x)\times(-2)$ (by chain rule)

$\therefore \int \cos(1-2x)\,dx = -\frac{1}{2}\sin(1-2x)+c$

e $\int \operatorname{cosec}^2 3x\,dx = -\frac{1}{3}\cot 3x + c$

f $\int \sec 4x \tan 4x\,dx = \frac{1}{4}\sec 4x + c$

g $\int 3\sin\left(\frac{1}{2}x+1\right)dx = -6\cos\left(\frac{1}{2}x+1\right)+c$

h $\int \sec^2(2-x)\,dx = -\tan(2-x)+c$

OR Let $y=\tan(2-x)$

then $\frac{dy}{dx}=\sec^2(2-x)\times(-1)$ (by chain rule)

$\therefore \int \sec^2(2-x)\,dx = -\tan(2-x)+c$

i $\int \operatorname{cosec} 2x \cot 2x\,dx = -\frac{1}{2}\operatorname{cosec} 2x + c$

j $\int (\cos 3x - \sin 3x)\,dx$

$= \frac{1}{3}\sin 3x + \frac{1}{3}\cos 3x + c$

$= \frac{1}{3}(\sin 3x + \cos 3x) + c$

2 a $\int \left(e^{2x} - \frac{1}{2}\sin(2x-1)\right)dx$

$= \frac{1}{2}e^{2x} + \frac{1}{4}\cos(2x-1)+c$

b $\int (e^x+1)^2\,dx$

$= \int (e^{2x}+2e^x+1)\,dx$

$= \frac{1}{2}e^{2x}+2e^x+x+c$

c $\int \sec^2 2x(1+\sin 2x)\,dx$

$= \int (\sec^2 2x + \sec^2 2x \sin 2x)\,dx$

$= \int (\sec^2 2x + \sec 2x \tan 2x)\,dx$

$= \frac{1}{2}\tan 2x + \frac{1}{2}\sec 2x + c$

d $\int \frac{3-2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}dx$

$= \int \left(3\operatorname{cosec}^2 \frac{1}{2}x - 2\operatorname{cosec}\frac{1}{2}x\cot\frac{1}{2}x\right)dx$

$= -6\cot\left(\frac{1}{2}x\right) + 4\operatorname{cosec}\left(\frac{1}{2}x\right)+c$

2 e $\int (e^{3-x}+\sin(3-x)+\cos(3-x))\,dx$

$= -e^{3-x}+\cos(3-x)-\sin(3-x)+c$

Note: extra minus signs from $-x$ terms and chain rule.

3 a $\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1|+c$

b $\int \frac{1}{(2x+1)^2}\,dx$

$= \int (2x+1)^{-2}\,dx$

$= \frac{(2x+1)^{-1}}{-1}\times\frac{1}{2}+c$

$= -\frac{1}{2(2x+1)}+c$

c $\int (2x+1)^2\,dx$

$= \frac{(2x+1)^3}{3} \times \frac{1}{2} + c$

$= \frac{(2x+1)^3}{6} + c$

d $\int \frac{3}{4x-1}\,dx = \frac{3}{4}\ln|4x-1| + c$

e $\int \frac{3}{1-4x}\,dx$

$= -\int \frac{3}{4x-1}\,dx$

$= -\frac{3}{4}\ln|4x-1| + c$

OR Let $y = \ln|1-4x|$

then $\frac{dy}{dx} = \frac{1}{1-4x} \times (-4)$ (by chain rule)

$\therefore \int \frac{3}{1-4x}\,dx = -\frac{3}{4}\ln|1-4x| + c$

Note: $\ln|1-4x| = \ln|4x-1|$ because of $|\ |$ sign.

f $\int \frac{3}{(1-4x)^2}\,dx$

$= \int 3(1-4x)^{-2}\,dx$

$= \frac{3}{-4} \times \frac{(1-4x)^{-1}}{-1} + c$

$= \frac{3}{4(1-4x)} + c$

g $\int (3x+2)^5\,dx = \frac{(3x+2)^6}{18} + c$

h $\int \frac{3}{(1-2x)^3}\,dx = \frac{3}{-2} \times \frac{(1-2x)^{-2}}{-2} + c$

$= \frac{3}{4(1-2x)^2} + c$

OR Let $y = (1-2x)^{-2}$

then $\frac{dy}{dx} = -2(1-2x)^{-3} \times (-2)$
(by chain rule)

$\therefore \int \frac{3}{(1-2x)^3}\,dx = \frac{3}{4}(1-2x)^{-2} + c$

4 a $\int \left(3\sin(2x+1) + \frac{4}{2x+1}\right)dx$

$= -\frac{3}{2}\cos(2x+1) + \frac{4}{2}\ln|2x+1| + c$

$= -\frac{3}{2}\cos(2x+1) + 2\ln|2x+1| + c$

b $\int \left(e^{5x} + (1-x)^5\right)dx = \int e^{5x}\,dx + \int (1-x)^5\,dx$

$= \frac{1}{5}e^{5x} - \frac{1}{6}(1-x)^6 + c$

OR Let $y = (1-x)^6$

then $\frac{dy}{dx} = 6(1-x)^5 \times (-1)$ (by chain rule)

$\therefore \int (1-x)^5\,dx = -\frac{1}{6}(1-x)^6 + c$

c $\int \left(\frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2}\right)dx$

$= \int \left(\operatorname{cosec}^2 2x + \frac{1}{1+2x} + (1+2x)^{-2}\right)dx$

$= -\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1+2x|$

$+ \frac{(1+2x)^{-1}}{-1} \times \frac{1}{2} + c$

$= -\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1+2x| - \frac{1}{2(1+2x)} + c$

d $\int\left((3x+2)^2+\frac{1}{(3x+2)^2}\right)\mathrm{d}x$

$=\int\left((3x+2)^2+(3x+2)^{-2}\right)\mathrm{d}x$

$=\frac{(3x+2)^3}{9}-\frac{(3x+2)^{-1}}{3}+c$

$=\frac{(3x+2)^3}{9}-\frac{1}{3(3x+2)}+c$

5 a $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos(\pi-2x)\mathrm{d}x=\left[-\frac{1}{2}\sin(\pi-2x)\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$=\left(-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)\right)-\left(-\frac{1}{2}\sin\frac{\pi}{2}\right)$

$=\frac{1}{2}+\frac{1}{2}=1$

b $\int_{\frac{1}{2}}^{1}\frac{12}{(3-2x)^4}\mathrm{d}x$

Consider $y=\frac{1}{(3-2x)^3}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{6}{(3-2x)^4}$

So $\int_{\frac{1}{2}}^{1}\frac{12}{(3-2x)^4}\mathrm{d}x=\left[\frac{2}{(3-2x)^3}\right]_{\frac{1}{2}}^{1}$

$=2-\frac{1}{4}=\frac{7}{4}$

c $\int_{\frac{2\pi}{9}}^{\frac{5\pi}{18}}\sec^2(\pi-3x)\mathrm{d}x=\left[-\frac{1}{3}\tan(\pi-3x)\right]_{\frac{2\pi}{9}}^{\frac{5\pi}{18}}$

$=\left(-\frac{1}{3}\tan\left(\pi-\frac{15\pi}{18}\right)\right)-\left(-\frac{1}{3}\tan\left(\pi-\frac{6\pi}{9}\right)\right)$

$=\left(-\frac{1}{3}\tan\frac{\pi}{6}\right)-\left(-\frac{1}{3}\tan\frac{\pi}{3}\right)$

$=\left(-\frac{1}{3}\times\frac{1}{\sqrt{3}}\right)-\left(-\frac{1}{3}\times\sqrt{3}\right)$

$=-\frac{\sqrt{3}}{9}+\frac{\sqrt{3}}{3}=\frac{2\sqrt{3}}{9}$

d $\int_2^3\frac{5}{7-2x}\mathrm{d}x=\left[-\frac{5}{2}\ln|7-2x|\right]_2^3$

$=\left(-\frac{5}{2}\ln 1\right)-\left(-\frac{5}{2}\ln 3\right)$

$=\frac{5}{2}\ln 3$

6 $\int_3^b(2x-6)^2\mathrm{d}x=\int_3^b(4x^2-24x+36)\mathrm{d}x$

$\left[\frac{4x^3}{3}-12x^2+36x\right]_3^b=36$

$\left(\frac{4b^3}{3}-12b^2+36b\right)-(36-108+108)=36$

$\frac{4b^3}{3}-12b^2+36b-72=0$

$b^3-9b^2+27b-54=0$

$(b-6)(b^2-3b+9)=0$

$b=6$ since $b^2-3b+9>0$.

7 $\int_{e^2}^{e^8}\frac{\mathrm{d}x}{kx}=\left[\frac{1}{k}\ln x\right]_{e^2}^{e^8}=\frac{1}{4}$

$\frac{8}{k}-\frac{2}{k}=\frac{1}{4}$

$k=32-8=24$

8 $\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}}(1-\pi\sin kx)\mathrm{d}x=\left[x+\frac{\pi}{k}\cos kx\right]_{\frac{\pi}{4k}}^{\frac{\pi}{3k}}$

$=\left(\frac{\pi}{3k}+\frac{\pi}{k}\cos\frac{\pi}{3}\right)-\left(\frac{\pi}{4k}+\frac{\pi}{k}\cos\frac{\pi}{4}\right)$

$=\left(\frac{\pi}{3k}+\frac{\pi}{2k}\right)-\left(\frac{\pi}{4k}+\frac{\pi}{\sqrt{2}k}\right)$

$=\frac{\pi}{k}\left(\frac{1}{3}+\frac{1}{2}\right)-\left(\frac{1}{4}+\frac{1}{\sqrt{2}}\right)=\frac{\pi}{k}\left(\frac{7}{12}-\frac{\sqrt{2}}{2}\right)$

$\frac{\pi}{k}\left(\frac{7}{12}-\frac{\sqrt{2}}{2}\right)=\pi(7-6\sqrt{2})$

$\frac{\pi}{k}\left(\frac{7-6\sqrt{2}}{12}\right)=\pi(7-6\sqrt{2})$

$k=\frac{1}{12}$

Challenge

$$\int_5^{11} \frac{1}{ax+b}\,dx = \left[\frac{1}{a}\ln|ax+b| + \frac{1}{a}\ln k\right]_5^{11}$$

where $\frac{1}{a}\ln k$ is a constant

$$= \frac{1}{a}\left[\ln k|ax+b|\right]_5^{11}$$

$$= \frac{1}{a}(\ln k|11a+b| - \ln k|5a+b|)$$

$$= \frac{1}{a}(\ln k|11a+b| - k\ln|5a+b|)$$

So $\ln k|11a+b| - \ln k|5a+b| = \ln\left(\frac{41}{17}\right)$

$$\ln\left|\frac{11a+b}{5a+b}\right| = \ln\left(\frac{41}{17}\right)$$

$$\frac{11a+b}{5a+b} = \pm\frac{41}{17}$$

Case 1:

$$\frac{11a+b}{5a+b} = \frac{41}{17}$$

$187a+17b = 205a+41b$

$18a = -24b$

$3a = -4b$

So a must be a multiple of 4 between 0 and 10.

$a = 4 \Rightarrow b = -3$

$a = 8 \Rightarrow b = -6$

Case 2:

$$\frac{11a+b}{5a+b} = -\frac{41}{17}$$

$187a+17b = -205a-41b$

$392a = -58b$

$b = -\frac{196}{29}a$

But this cannot be an integer, since $a < 29$, so case 2 gives no possible solutions.

Therefore the only two possible solutions are $a = 4$, $b = -3$ and $a = 8$, $b = -6$.