Numerical methods 10D

1 a $M = E - 0.1\sin E$

When $M = \frac{\pi}{6}$, $\frac{\pi}{6} = E - 0.1\sin E$

$f(x) = x - 0.1\sin x - k$

If E is a root of $f(x)$

$f(E) = E - 0.1\sin E - k = 0$

$k = \frac{\pi}{6}$

b $f(E) = E - 0.1\sin E - \frac{\pi}{6}$

$f'(E) = 1 - 0.1\cos E$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_0 = 0.6 \Rightarrow$

$x_1 = 0.6 - \frac{0.6 - 0.1\times\sin 0.6 - \frac{\pi}{6}}{1 - 0.1\cos 0.6}$

$= 0.6 - \frac{0.019937...}{0.91746...} = 0.5782...$

c $f(0.5775) = 0.5775 - 0.1\times\sin 0.5775 - \frac{\pi}{6}$

$= -0.00069...$

$f(0.5785) = 0.5785 - 0.1\times\sin 0.5785 - \frac{\pi}{6}$

$= 0.00022...$

There is a change of sign in this interval so $E = 0.578$ correct to 3 d.p.

2 a At A and B, $v = 0$.

$v = 0 \Rightarrow \left(10 - \frac{1}{2}(t+1)\right)\ln(t+1) = 0$

$\ln(t+1) = 0 \Rightarrow t = 0$

$10 - \frac{1}{2}(t+1) = 0 \Rightarrow t = 19$

So A is (0, 0) and B is (19, 0).

b $f'(t) = \left(10 - \frac{1}{2}(t+1)\right)\times\frac{1}{t+1} - \frac{1}{2}\ln(t+1)$

$f'(t) = \frac{10}{t+1} - \frac{1}{2}\left(\ln(t+1) + 1\right)$

c $f'(5.8) = \frac{10}{5.8+1} - 0.5(\ln(5.8+1) + 1)$

$= 0.0121...$

$f'(5.9) = \frac{10}{5.9+1} - 0.5(\ln(5.9+1) + 1)$

$= -0.0164...$

The sign of the gradient changes in the interval [5.8, 5.9] so the x-coordinate of P is in this interval.

d At the stationary point $f'(t) = 0$.

$\frac{10}{t+1} - \frac{1}{2}(\ln(t+1) + 1) = 0$

$\frac{10}{t+1} = \frac{1}{2}(\ln(t+1) + 1)$

$\frac{20}{\ln(t+1) + 1} = t + 1$

$t = \frac{20}{1 + \ln(t+1)} - 1$

e $t_1 = \frac{20}{1 + \ln(t_0 + 1)} - 1 = \frac{20}{1 + \ln 6} - 1 = 6.1639$

$t_2 = \frac{20}{1 + \ln 7.1639} - 1 = 5.7361$

$t_3 = \frac{20}{1 + \ln 6.7361} - 1 = 5.8787$

To 3 d.p. the values are $t_1 = 6.164$, $t_2 = 5.736$ and $t_3 = 5.879$.

3 a $d(x) = e^{-0.6x}(x^2 - 3x)$

$d(x) = 0 \Rightarrow x^2 - 3x = 0$

$x(x-3) = 0 \Rightarrow x = 0$ or 3

The stream is 3 metres wide so the function is only valid for $0 \leqslant x \leqslant 3$.

b $d'(x) = e^{-0.6x}(2x - 3) - \frac{3}{5}e^{-0.6x}(x^2 - 3x)$

$= 2xe^{-0.6x} - 3e^{-0.6x} - \frac{3}{5}x^2e^{-0.6x} + \frac{9}{5}xe^{-0.6x}$

$= e^{-0.6x}\left(-\frac{3}{5}x^2 + \frac{19}{5}x - \frac{15}{5}\right)$

$d'(x) = -\frac{1}{5}e^{-0.6x}(3x^2 - 19x + 15)$

So $a = 3$, $b = -19$, $c = 15$.

3 c i $-\frac{1}{5}e^{-0.6x}(3x^2-19x+15)=0$

$-\frac{1}{5}e^{-0.6x}\neq 0$

so $d'(x)=0\Rightarrow 3x^2-19x+15=0$

$3x^2=19x-15$

$x=\sqrt{\frac{19x-15}{3}}$

ii $3x^2-19x+15=0$

$19x=3x^2+15$

$x=\frac{3x^2+15}{19}$

iii $3x^2=19x-15$

$x=\frac{19x-15}{3x}$

d For $x_0=1$ in equation from **c i**
Iterates to 5.409 after 21 iterations.

For $x_0=1$ in equation from **c iii**
Iterates to 5.409 after 8 iterations.

These are both outside the required range.

For $x_0=1$ in equation from **c ii**
Iterates to 0.924 after 6 iterations.

e $d(0.924)=e^{-0.6\times0.924}(0.924^2-3\times0.924)$
$=-1.1018...$

The maximum depth of the river is
1.10 m, correct to 2 d.p.

4 a $h(t)=40\sin\left(\frac{t}{10}\right)-9\cos\left(\frac{t}{10}\right)-0.5t^2+9$

$h(t)=0\Rightarrow$

$40\sin\left(\frac{t}{10}\right)-9\cos\left(\frac{t}{10}\right)-0.5t^2+9=0$

$0.5t^2=40\sin\left(\frac{t}{10}\right)-9\cos\left(\frac{t}{10}\right)+9$

$t^2=18+80\sin\left(\frac{t}{10}\right)-18\cos\left(\frac{t}{10}\right)$

$t=\sqrt{18+80\sin\left(\frac{t}{10}\right)-18\cos\left(\frac{t}{10}\right)}$

b $t_1=\sqrt{18+80\sin\left(\frac{8}{10}\right)-18\cos\left(\frac{8}{10}\right)}$

$t_1=7.928$

$t_2=7.896$

$t_3=7.882$

$t_4=7.876$

c $h'(t)=4\cos\left(\frac{t}{10}\right)+\frac{9}{10}\sin\left(\frac{t}{10}\right)-t$

d $h(8)=40\sin 0.8-9\cos 0.8-32+9$
$=-0.5761$

$h'(8)=4\cos 0.8+0.9\sin 0.8-8$
$=-4.5676$

Second approximation:
$=8-\frac{h(8)}{h'(8)}=8-\frac{-0.5761}{-4.5676}=7.874$ to 3 d.p.

e Restrict the range of validity to $0\leqslant t\leqslant A$.

5 a $c(x)=5e^{-x}+4\sin\left(\frac{x}{2}\right)+\frac{x}{2}$

$c'(x)=-5e^{-x}+2\cos\left(\frac{x}{2}\right)+\frac{1}{2}$

b Turning points are when $c'(x)=0$

$-5e^{-x}+2\cos\left(\frac{x}{2}\right)+\frac{1}{2}=0$

i $2\cos\left(\frac{x}{2}\right)=5e^{-x}-\frac{1}{2}$

$\cos\left(\frac{x}{2}\right)=\frac{5}{2}e^{-x}-\frac{1}{4}$

$x=2\arccos\left(\frac{5}{2}e^{-x}-\frac{1}{4}\right)$

5 b ii $5e^{-x} = 2\cos\left(\frac{x}{2}\right) + \frac{1}{2}$

$$5e^{-x} = \frac{4\cos\left(\frac{x}{2}\right)+1}{2}$$

$$10e^{-x} = 4\cos\left(\frac{x}{2}\right)+1$$

$$e^{-x} = \frac{4\cos\left(\frac{x}{2}\right)+1}{10}$$

$$e^{x} = \frac{10}{4\cos\left(\frac{x}{2}\right)+1}$$

$$x = \ln\left(\frac{10}{4\cos\left(\frac{x}{2}\right)+1}\right)$$

c $x_1 = 2\arccos\left(\frac{5}{2}e^{-3} - \frac{1}{4}\right) = 3.393$

$$x_2 = 2\arccos\left(\frac{5}{2}e^{-3.393} - \frac{1}{4}\right) = 3.475$$

$$x_3 = 2\arccos\left(\frac{5}{2}e^{-3.475} - \frac{1}{4}\right) = 3.489$$

$$x_4 = 2\arccos\left(\frac{5}{2}e^{-3.489} - \frac{1}{4}\right) = 3.491$$

d $x_1 = \ln\left(\frac{10}{4\cos\left(\frac{1}{2}\right)+1}\right) = 0.796$

$$x_2 = \ln\left(\frac{10}{4\cos\left(\frac{0.796}{2}\right)+1}\right) = 0.758$$

$$x_3 = \ln\left(\frac{10}{4\cos\left(\frac{0.758}{2}\right)+1}\right) = 0.752$$

$$x_3 = \ln\left(\frac{10}{4\cos\left(\frac{0.752}{2}\right)+1}\right) = 0.751$$

e The model does support the assumption that the crime rate was increasing. The model shows that there is a minimum point 3/4 of the way through 2000 and a maximum point mid-way through 2003. So, the crime rate is increasing in the interval between October 2000 and June 2003.