Numerical methods 10C

1 a $f(x) = x^3 - 2x - 1$

$f(1) = -2$

$f(2) = 3$

There is a change of sign, so there is a root α in the interval [1, 2].

b $f(x) = x^3 - 2x - 1$

$f'(x) = 3x^2 - 2$

Using $x_0 = 1.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.5 - \frac{(-0.625)}{4.75}$$

$$x_1 = 1.6316$$

$x_1 = 1.632$ correct to 3 d.p.

2 a $f(x) = x^2 - \frac{4}{x} + 6x - 10$

$$f'(x) = 2x + \frac{4}{x^2} + 6 = 2\left(x + \frac{2}{x^2} + 3\right)$$

b $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Using $x_0 = -0.4$

$$x_1 = -0.4 - \frac{0.4^2 - \frac{4}{-0.4} + 6\times(-0.4) - 10}{2\left(-0.4 + \frac{2}{-0.4^2} + 3\right)}$$

$$= -0.4 - \frac{-2.24}{30.2}$$

$$= -0.4 + 0.07417...$$

$$= -0.3258...$$

$x_1 = -0.326$ correct to 3 d.p.

3 a $f(x) = x^{\frac{3}{2}} - e^{-x} + \frac{1}{\sqrt{x}} - 2$

A is a stationary point on the curve so $f'(q) = 0$. It is not possible to divide by zero using the Newton–Raphson method, so this value of x_0 cannot be used.

b $f'(x) = \frac{3}{2}x^{\frac{1}{2}} + e^{-x} - \frac{1}{2x^{\frac{3}{2}}}$

$$f(1.2) = 1.2^{\frac{3}{2}} - e^{-1.2} + \frac{1}{\sqrt{1.2}} - 2$$

$$= -0.07389...$$

$$f'(1.2) = \frac{3}{2}\sqrt{1.2} + e^{-1.2} - \frac{1}{2(1.2)^{\frac{3}{2}}}$$

$$= 1.56399...$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow$$

$$x_1 = 1.2 + \frac{0.07389...}{1.56399...} = 1.247 \text{ to 3 d.p.}$$

4 a $f(x) = 1 - x - \cos(x^2)$

$f(1.4) = 1 - 1.4 - \cos(1.4)^2 = -0.0205...$

$f(1.5) = 1 - 1.5 - \cos(1.5)^2 = 0.128...$

There is a change of sign in the interval [1.4, 1.5] so there must be a root α in this interval.

b $f'(x) = -1 + 2x\sin(x^2)$

$f'(1.4) = -1 + 2.8\sin 1.96 = 1.5905...$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow$$

$$x_1 = 1.4 + \frac{0.0205...}{1.5905...} = 1.413 \text{ correct to 3 d.p.}$$

c $f(1.4125) = -0.00076... < 0$

$f(1.4135) = 0.00081... > 0$

There is a sign change in the interval [1.4125, 1.4135] so $x = 1.413$ is correct to 3 d.p.

5 a $f(x) = x^2 - \frac{3}{x^2}$

$$f(1.3) = 1.69 - \frac{3}{1.69} = -0.0851...$$

$$f(1.4) = 1.96 - \frac{3}{1.96} = 0.429...$$

There is a change of sign in the interval [1.3, 1.4] so there must be a root α in this interval.

5 b $f'(x) = 2x + \frac{6}{x^3}$

c $f'(1.3) = 2.6 + \frac{6}{1.3^3} = 5.3309...$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow$

$x_1 = 1.3 + \frac{0.0851...}{5.3309...} = 1.316$ to 3 d.p.

6 a $f(x) = x^2 \sin x - 2x + 1$

i $f(0.6) = 0.36\sin 0.6 - 1.2 + 1$
$= 0.0032...$
$f(0.7) = 0.49\sin 0.7 - 1.4 + 1$
$= -0.0843...$

ii $f(1.2) = 1.44\sin 1.2 - 2.4 + 1$
$= -0.0578...$
$f(1.3) = 1.69\sin 1.3 - 2.6 + 1$
$= 0.0284...$

iii $f(2.4) = 5.76\sin 2.4 - 4.8 + 1$
$= 0.0906...$
$f(2.5) = 6.25\sin 2.5 - 5 + 1$
$= -0.2595...$

There is a change of sign in all the intervals so there must be a root in each.

b There is a stationary point at $x = a$, so $f'(x) = 0$ here. You cannot divide by zero in the Newton–Raphson formula so $x_0 = a$ cannot be used as a first approximation.

c $f'(x) = x^2 \cos x + 2x\sin x - 2$

$f'(2.4) = 5.76\cos 2.4 + 4.8\sin 2.4 - 2$
$= -3.0051...$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow$

$x_1 = 2.4 + \frac{0.0906...}{3.0051...} = 2.430$ to 3 d.p.

7 a $f(x) = \ln(3x-4) - x^2 + 10$

$f(3.4) = \ln 6.2 - 11.56 + 10 = 0.2645...$

$f(3.5) = \ln 6.5 - 12.25 + 10 = -0.3781...$

There is a change of sign in the interval [3.4, 3.5] so there must be a root α in this interval.

b $f'(x) = \frac{3}{3x-4} - 2x$

c $f'(3.4) = \frac{3}{6.2} - 6.8 = -6.3161...$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow$

$x_1 = 3.4 + \frac{0.2645...}{6.3161...} = 3.442$ to 3 d.p.

Challenge

a From the graph, $f(x) > 0$ for all values of $x > 0$. Note also that $xe^{-x^2} > 0$ when $x > 0$.

So the same must be true for $x > \frac{1}{\sqrt{2}}$.

$f'(x) = e^{-x^2}(1 - 2x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$

So $f'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$

$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$ is an increasing sequence as $f(x) > 0$ and $f'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$.

Therefore the Newton–Raphson method will fail to converge.

Challenge

b $f(-0.5) = \frac{1}{5} + (-0.5)e^{-0.25} = -0.1894...$

$f'(-0.5) = e^{-0.25}(1 - 0.5) = 0.3894...$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow$

$x_1 = -0.5 + \frac{0.1894...}{0.3894...} = -0.0136...$

$x_2 = -0.0136... + \frac{0.1864...}{0.9994...} = -0.2001...$

$x_3 = -0.2001... - \frac{0.0077...}{0.8838...} = -0.2088...$

$x_4 = -0.2088... - \frac{0.0000...}{0.8737...} = -0.2089...$

The root is −0.209 correct to 3 d.p.