

Numerical methods 10B

1 a i $x^2 - 6x + 2 = 0$

$6x = x^2 + 2$ Add 6x to each side.

$x = \frac{x^2 + 2}{6}$ Divide each side by 6.

ii $x^2 - 6x + 2 = 0$

$x^2 + 2 = 6x$ Add 6x to each side.

$x^2 = 6x - 2$ Subtract 2 from each side.

$x = \sqrt{6x - 2}$ Take the square root of each side.

iii $x^2 - 6x + 2 = 0$

$x^2 + 2 = 6x$ Add 6x to each side.

$x^2 = 6x - 2$ Subtract 2 from each side.

$\frac{x^2}{x} = \frac{6x}{x} - \frac{2}{x}$ Divide each term by x.

$x = 6 - \frac{2}{x}$ Simplify.

b i $x_0 = 4 \Rightarrow x_1 = \frac{4^2 + 2}{6} = 3$

$x_2 = \frac{3^2 + 2}{6} = 1.83333$

$x_3 = \frac{1.83333^2 + 2}{6} = 0.89352$

$x_4 = \frac{0.89352^2 + 2}{6} = 0.46640$

$x_5 = \frac{0.46640^2 + 2}{6} = 0.36959$

$x_6 = \frac{0.36959^2 + 2}{6} = 0.35610$

$x_7 = \frac{0.35610^2 + 2}{6} = 0.35447$

$x_8 = \frac{0.35447^2 + 2}{6} = 0.35428$

$x = 0.354$ to 3 d.p.

ii $x_0 = 4 \Rightarrow x_1 = \sqrt{6x - 2} = 4.69042$

$x_2 = \sqrt{6 \times 4.69042 - 2} = 5.11297$

⋮

$x_{15} = \sqrt{6 \times 5.64547 - 2} = 5.6456$

$x = 5.646$ to 3 d.p.

iii $x_0 = 4 \Rightarrow x_1 = 6 - \frac{2}{x} = 5.5$

$x_2 = 5.5 - \frac{2}{x} = 5.63636$

$x_3 = 5.63636 - \frac{2}{x} = 5.64516$

$x_4 = 5.64516 - \frac{2}{x} = 5.64571$

$x = 5.646$ to 3 d.p.

c $x^2 - 6x + 2 = 0$

$x = \frac{6 \pm \sqrt{36 - 8}}{2} = 3 \pm \sqrt{7}$

$a = 3, b = 7$

2 a i $f(x) = 0$

$x^2 - 5x - 3 = 0$

$x^2 = 5x + 3$ Add 5x + 3 to each side.

$x = \sqrt{5x + 3}$ Take the square root of each side.

ii $f(x) = 0$

$x^2 - 5x - 3 = 0$

$5x = x^2 - 3$ Add 5x to each side.

$x = \frac{x^2 - 3}{5}$ Divide each side by 5.

b i $x_0 = 5 \Rightarrow x_1 = \sqrt{5 \times 5 + 3} = \sqrt{28} = 5.2915$

$x_2 = \sqrt{5 \times 5.2915 + 3} = 5.4275$

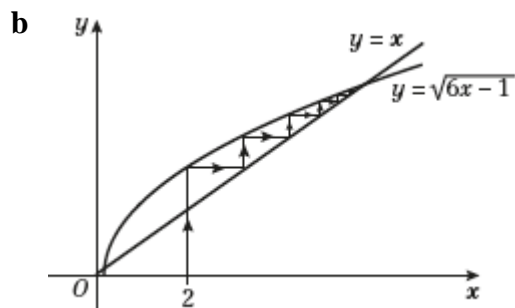
$x_3 = \sqrt{5 \times 5.4275 + 3} = 5.4900$

$x_4 = \sqrt{5 \times 5.4900 + 3} = 5.5180$

$x = 5.5$ to 1 d.p.

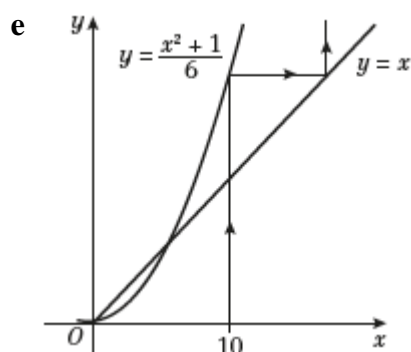
2 b ii $x_0 = 5 \Rightarrow x_1 = \frac{5^2 - 3}{5} = 4.4$
 $x_2 = \frac{4.4^2 - 3}{5} = 3.272$
 $x_3 = \frac{3.272^2 - 3}{5} = 1.5412$
 $x_4 = \frac{1.5412^2 - 3}{5} = -0.1249$
 $x_5 = \frac{(-0.1249)^2 - 3}{5} = -0.5969$
 $x_6 = \frac{(-0.5969)^2 - 3}{5} = -0.5287$
 $x = -0.5$ to 1 d.p.

3 a $f(x) = 0$
 $x^2 - 6x + 1 = 0$
 $x^2 = 6x - 1$ Add $6x - 1$ to each side.
 $x = \sqrt{6x - 1}$ Take the square root of each side.



c The curve crosses the line twice so there are two roots.

d See diagram above.



4 a $f(x) = xe^{-x} - x + 2$
 $f(x) = 0 \Rightarrow e^{-x} = \frac{x-2}{x}$
 $e^x = \frac{x}{x-2}$
 $x = \ln \left| \frac{x}{x-2} \right|, x \neq 2$

b $x_0 = -1 \Rightarrow x_1 = \ln \left| \frac{-1}{-3} \right| = -1.0986$
 $x_2 = \ln \left| \frac{-1.0986}{-1.0986 - 2} \right| = -1.0369$
 $x_3 = \ln \left| \frac{-1.0369}{-1.0369 - 2} \right| = -1.0746$

To 2 d.p., $x_1 = -1.10$, $x_2 = -1.04$,
 $x_3 = -1.07$.

5 a i $f(x) = 0$
 $x^3 + 5x^2 - 2 = 0$
 $x^3 + 5x^2 = 2$ Add 2 to each side.
 $x^3 = 2 - 5x^2$ Subtract $5x^2$ from each side.
 $x = \sqrt[3]{2 - 5x^2}$ Take the cube root of each side.

ii $f(x) = 0$
 $x^3 + 5x^2 - 2 = 0$
 $x^3 + 5x^2 = 2$ Add 2 to each side.
 $x^3 = 2 - 5x^2$ Subtract $5x^2$ from each side.

$\frac{x^3}{x^2} = \frac{2}{x^2} - \frac{5x^2}{x^2}$ Divide each term by x^2 .
 $x = \frac{2}{x^2} - 5$ Simplify.

5 a iii $f(x) = 0$

$$x^3 + 5x^2 - 2 = 0$$

$$x^3 + 5x^2 = 2 \quad \text{Add 2 to each side.}$$

$$5x^2 = 2 - x^3 \quad \text{Subtract } x^3 \text{ from each side.}$$

$$x^2 = \frac{2 - x^3}{5} \quad \text{Divide each side by 5.}$$

$$x = \sqrt{\frac{2 - x^3}{5}} \quad \text{Take the square root of each side.}$$

b $x_0 = 10 \Rightarrow x_1 = \frac{2}{10^2} - 5 = -4.98$

$$x_2 = \frac{2}{4.98^2} - 5 = -4.9194$$

$$x_3 = \frac{2}{4.9194^2} - 5 = -4.9174$$

$$x_4 = \frac{2}{4.9174^2} - 5 = -4.9173$$

$$x = -4.917 \text{ to 3 d.p.}$$

c $x_0 = 1 \Rightarrow x_1 = \sqrt{\frac{2 - 1^3}{5}} = 0.4472$

$$x_2 = \sqrt{\frac{2 - 0.4472^3}{5}} = 0.6182$$

$$x_3 = \sqrt{\frac{2 - 0.6182^3}{5}} = 0.5939$$

$$x_4 = \sqrt{\frac{2 - 0.5939^3}{5}} = 0.5984$$

$$x_5 = \sqrt{\frac{2 - 0.5984^3}{5}} = 0.5976$$

$$x_6 = \sqrt{\frac{2 - 0.5976^3}{5}} = 0.5978$$

$$x = 0.598 \text{ to 3 d.p.}$$

d $x_0 = 2 \Rightarrow x_1 = \sqrt{\frac{2 - 2^3}{5}} = \sqrt{-\frac{6}{5}}$

This has no real number solutions so no iteration is possible.

6 a $x^4 - 3x^3 - 6 = 0$

$$3x^3 = x^4 - 6 \quad \text{Add } 3x^3 \text{ to each side.}$$

$$\frac{3x^3}{3} = \frac{x^4}{3} - \frac{6}{3} \quad \text{Divide each term by 3.}$$

$$x^3 = \frac{x^4}{3} - 2 \quad \text{Simplify.}$$

$$x = \sqrt[3]{\frac{x^4}{3} - 2} \quad \text{Take the cube root of each side.}$$

So $p = \frac{1}{3}$ and $q = -2$.

b $x_0 = 0 \Rightarrow x_1 = \sqrt[3]{\frac{1}{3} \cdot 0^4 - 2} = -1.25992$

$$x_2 = \sqrt[3]{\frac{1}{3} \times 1.25992^4 - 2} = -1.05073$$

$$x_3 = \sqrt[3]{\frac{1}{3} \times 1.05073^4 - 2} = -1.16807$$

To 3 d.p., $x_1 = -1.260$, $x_2 = -1.051$, $x_3 = -1.168$.

c $f(-1.1315)$
 $= (-1.1315)^4 - 3 \times (-1.1315)^3 - 6$
 $= -0.0148\dots$

$f(-1.1325)$
 $= (-1.1325)^4 - 3 \times (-1.1325)^3 - 6$
 $= -0.0024\dots$

There is a change of sign in this interval so $\alpha = 1.132$ to 3 d.p.

7 a $f(x) = 3\cos(x^2) + x - 2$
 $f(x) = 0 \Rightarrow 3\cos(x^2) = 2 - x$

$$\cos(x^2) = \frac{2-x}{3}$$

$$x^2 = \arccos\left(\frac{2-x}{3}\right)$$

$$x = \left(\arccos\left(\frac{2-x}{3}\right)\right)^{\frac{1}{2}}$$

b $x_0 = 1 \Rightarrow x_1 = \left(\arccos\left(\frac{2-1}{3}\right)\right)^{\frac{1}{2}} = 1.1094$

$$x_2 = \left(\arccos\left(\frac{2-1.1094}{3}\right)\right)^{\frac{1}{2}} = 1.1267$$

$$x_3 = \left(\arccos\left(\frac{2-1.1267}{3}\right)\right)^{\frac{1}{2}} = 1.1293$$

To 3 d.p., $x_1 = 1.109$, $x_2 = 1.127$,
 $x_3 = 1.129$.

c $f(1.12975)$
 $= 3\cos(1.12975)^2 + 1.12975 - 2$
 $= 0.0004\dots$

$f(1.12985)$
 $= 3\cos(1.12985)^2 + 1.12985 - 2$
 $= -0.0001\dots$

There is a change of sign in this interval
 so $\alpha = 1.1298$ to 4 d.p.

8 a $f(x) = 4 \cot x - 8x + 3$
 $f(0.8) = 4 \cot 0.8 - 6.4 + 3 = 0.484\dots$
 $f(0.9) = 4 \cot 0.9 - 7.2 + 3 = -1.025\dots$

There is a change of sign in the interval
 $[0.8, 0.9]$, so there must be a root in this
 interval, since f is continuous over the
 interval.

b $f(x) = 0$
 $4 \cot x - 8x + 3 = 0$
 $8x = 4 \cot x + 3 = 4 \frac{\cos x}{\sin x} + 3$
 $x = \frac{\cos x}{2 \sin x} + \frac{3}{8}$

c $x_0 = 0.85 \Rightarrow x_1 = \frac{\cos 0.85}{2 \sin 0.85} + \frac{3}{8} = 0.8142$

$$x_2 = \frac{\cos 0.8142}{2 \sin 0.8142} + \frac{3}{8} = 0.8470$$

$$x_3 = \frac{\cos 0.8470}{2 \sin 0.8470} + \frac{3}{8} = 0.8169$$

d $f(0.8305) = 4 \cot 0.8305 - 6.644 + 3$
 $= 0.0105\dots$
 $f(0.8315) = 4 \cot 0.8315 - 6.652 + 3$
 $= -0.0047\dots$

There is a change of sign in this interval
 so $\alpha = 0.831$ to 3 d.p.

9 a $g(x) = e^{x-1} + 2x - 15$
 $g(x) = 0 \Rightarrow e^{x-1} = 15 - 2x$
 $x - 1 = \ln(15 - 2x), x < \frac{15}{2}$
 $x = \ln(15 - 2x) + 1, x < \frac{15}{2}$

b $x_0 = 3 \Rightarrow x_1 = \ln(15 - 2 \times 3) + 1 = 3.1972$
 $x_2 = \ln(15 - 2 \times 3.1972) + 1 = 3.1524$
 $x_3 = \ln(15 - 2 \times 3.1524) + 1 = 3.1628$

c $f(3.155) = e^{3.155-1} + 2 \times 3.155 - 15$
 $= -0.062\dots$
 $f(3.165) = e^{3.165-1} + 2 \times 3.165 - 15$
 $= 0.044\dots$

There is a change of sign in this interval
 so $\alpha = 3.16$ to 2 d.p.

10 a $f(x) = xe^x - 4x$

At A and B , $f(x) = 0$

$$xe^x - 4x = 0 \Rightarrow x(e^x - 4) = 0$$

$$x = 0 \text{ or } \ln 4$$

Coordinates of A and B are

$(0, 0)$ and $(\ln 4, 0)$.

b $f'(x) = xe^x + e^x - 4 = e^x(x+1) - 4$

c $f'(0.7) = e^{0.7}(0.7+1) - 4 = -0.5766\dots$

$$f'(0.8) = e^{0.8}(0.8+1) - 4 = 0.0059\dots$$

There is a change of sign in the interval, which implies $f'(x) = 0$, i.e. a stationary point, in this range.

d $f'(x) = 0 \Rightarrow e^x(x+1) - 4 = 0$

$$e^x = \frac{4}{x+1}$$

$$x = \ln\left(\frac{4}{x+1}\right)$$

e $x_0 = 0 \Rightarrow x_1 = \ln\left(\frac{4}{0+1}\right) = 1.3863$

$$x_2 = \ln\left(\frac{4}{1.3863+1}\right) = 0.5166$$

$$x_3 = \ln\left(\frac{4}{0.5166+1}\right) = 0.9699$$

$$x_4 = \ln\left(\frac{4}{0.9699+1}\right) = 0.7083$$

To 3 d.p., $x_1 = 1.386$, $x_2 = 0.517$,
 $x_3 = 0.970$, $x_4 = 0.708$.