

Numerical methods 10A

1 a $f(x) = x^3 - x + 5$

$$f(-2) = -8 + 2 + 5 = -1 < 0$$

$$f(-1) = -1 + 1 + 5 = 5 > 0$$

There is a change of sign between -2 and -1 so there is at least one root in the interval $-2 < x < -1$.

b $f(x) = x^2 - \sqrt{x} - 10$

$$f(3) = 9 - \sqrt{3} - 10 = -2.732... < 0$$

$$f(4) = 16 - \sqrt{4} - 10 = 4 > 0$$

There is a change of sign between 3 and 4 so there is at least one root in the interval $3 < x < 4$.

c $f(x) = x^3 - \frac{1}{x} - 2$

$$f(-0.5) = (-0.5)^3 + 2 - 2 = -0.125 < 0$$

$$f(-0.2) = (-0.2)^3 + 5 - 2 = 2.992 > 0$$

There is a change of sign between -0.5 and -0.2 so there is at least one root in the interval $-0.5 < x < -0.2$.

d $f(x) = e^x - \ln x - 5$

$$f(1.65) = e^{1.65} - \ln 1.65 - 5 = -0.293... < 0$$

$$f(1.75) = e^{1.75} - \ln 1.75 - 5 = 0.194... > 0$$

There is a change of sign between 1.65 and 1.75 so there is at least one root in the interval $1.65 < x < 1.75$.

2 a $f(x) = 3 + x^2 - x^3$

$$f(1.8) = 3 + 1.8^2 - 1.8^3 = 0.408 > 0$$

$$f(1.9) = 3 + 1.9^2 - 1.9^3 = -0.249 < 0$$

There is a change of sign so there is a root, α , in the interval $[1.8, 1.9]$.

b Choose interval $[1.8635, 1.8645]$ to test for root.

$$\begin{aligned} f(1.8635) &= 3 + 1.8635^2 - 1.8635^3 \\ &= 0.00138... > 0 \end{aligned}$$

$$\begin{aligned} f(1.8645) &= 3 + 1.8645^2 - 1.8645^3 \\ &= -0.00531... < 0 \end{aligned}$$

There is a change of sign between 1.8635 and 1.8645 , so $1.8635 < \alpha < 1.8645$, which gives $\alpha = 1.864$ correct to 3 d.p.

3 a $h(x) = \sqrt[3]{x} - \cos x - 1$

$$h(1.4) = \sqrt[3]{1.4} - \cos 1.4 - 1 = -0.0512... < 0$$

$$h(1.5) = \sqrt[3]{1.5} - \cos 1.5 - 1 = 0.0739... > 0$$

There is a change of sign so there is a root, α , in the interval $[1.4, 1.5]$.

b Choose interval $[1.4405, 1.4415]$ to test for root.

$$\begin{aligned} h(1.4405) &= \sqrt[3]{1.4405} - \cos 1.4405 - 1 \\ &= -0.00055... < 0 \end{aligned}$$

$$\begin{aligned} h(1.4415) &= \sqrt[3]{1.4415} - \cos 1.4415 - 1 \\ &= 0.00069... > 0 \end{aligned}$$

There is a change of sign between 1.4405 and 1.4415 , so $1.4405 < \alpha < 1.4415$, which gives $\alpha = 1.441$ correct to 3 d.p.

4 a $f(x) = \sin x - \ln x$

$$f(2.2) = \sin 2.2 - \ln 2.2 = 0.020... > 0$$

$$f(2.3) = \sin 2.3 - \ln 2.3 = -0.087... < 0$$

There is a change of sign so there is a root, α , in the interval $[2.2, 2.3]$.

b Choose interval $[2.2185, 2.2195]$ to test for root.

$$\begin{aligned} f(2.2185) &= \sin 2.2185 - \ln 2.2185 \\ &= 0.00064... > 0 \end{aligned}$$

$$\begin{aligned} f(2.2195) &= \sin 2.2195 - \ln 2.2195 \\ &= -0.00041... < 0 \end{aligned}$$

There is a change of sign between 2.2185 and 2.2195 , so $2.2185 < \alpha < 2.2195$, which gives $\alpha = 2.219$ correct to 3 d.p.

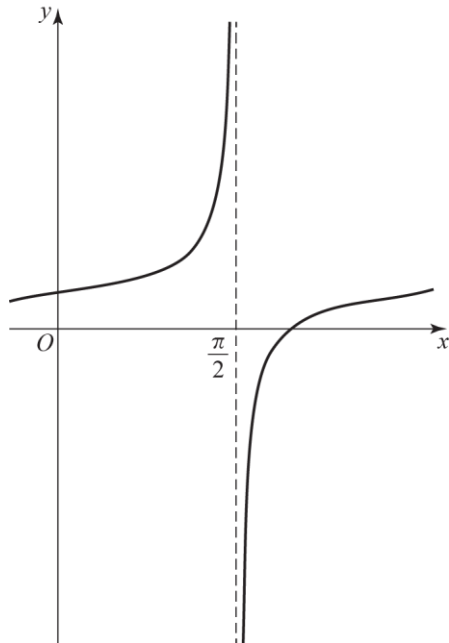
5 a $f(x) = 2 + \tan x$

$$f(1.5) = 2 + \tan 1.5 = 16.1... > 0$$

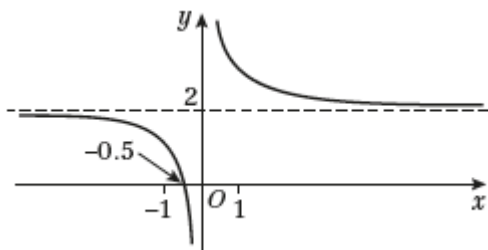
$$f(1.6) = 2 + \tan 1.6 = -32.2... < 0$$

So there is a change of sign in the interval $[1.5, 1.6]$.

- 5 b A sketch shows there is a vertical asymptote in the graph of $y = f(x)$ at $x = \frac{\pi}{2} = 1.57\dots$. So there is no root in the interval $[1.5, 1.6]$.



- 6 A sketch shows a root at -0.5 .



Or $f(x) = 0$ when $\frac{1}{x} + 2 = 0 \Rightarrow x = -\frac{1}{2}$
which is in the interval $[-1, 1]$.

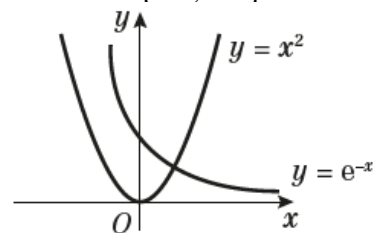
- 7 a $f(x) = (105x^3 - 128x^2 + 49x - 6)\cos 2x$
 $f(0.2) = (0.84 - 5.12 + 9.8 - 6)\cos 0.4$
 $= -0.442\dots < 0$
 $f(0.8) = (53.76 - 81.92 + 39.2 - 6)\cos 1.6$
 $= -0.147\dots < 0$

- b There is no sign change, so there are either no roots or an even number of roots in the interval $[0.2, 0.8]$.

- c $f(0.3) = (2.835 - 11.52 + 14.7 - 6)\cos 0.6$
 $= 0.0123\dots > 0$
 and...
 $f(0.4) = -0.111\dots < 0$
 $f(0.5) = -0.202\dots < 0$
 $f(0.6) = 0$
 $f(0.7) = 0.271\dots > 0$

- d From the changes in sign, there exists at least one root in each of the intervals $0.2 < x < 0.3$, $0.3 < x < 0.4$ and $0.7 < x < 0.8$. There is also a root at 0.6 . Therefore there are at least four roots in the interval $[0.2, 0.8]$.

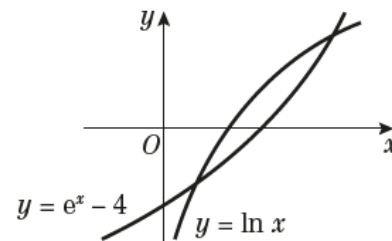
- 8 a



- b The curves meet where $e^{-x} = x^2$.
The curves meet at one point, so there is one value of x that satisfies the equation $e^{-x} = x^2$.
So $e^{-x} = x^2$ has one root.

- c $f(x) = e^{-x} - x^2$
 $f(0.70) = e^{-0.70} - 0.70^2 = 0.0065\dots$
 $f(0.71) = e^{-0.71} - 0.71^2 = 0.0124\dots$
 There is a change of sign between 0.70 and 0.71 so there is at least one root in the interval $0.70 < x < 0.71$.

- 9 a



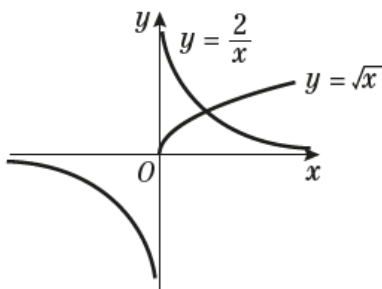
- b The curves meet at two points, so there are two values of x that satisfy the equation $\ln x = e^x - 4$. So $\ln x = e^x - 4$ has two roots.

- 9 c** $f(x) = \ln x - e^x + 4$
 $f(1.4) = \ln 1.4 - e^{1.4} + 4 = 0.281\dots$
 $f(1.5) = \ln 1.5 - e^{1.5} + 4 = -0.0762\dots$
 There is a change of sign between 1.4 and 1.5 so there is at least one root in the interval $1.4 < x < 1.5$.

- 10 a** $h(x) = \sin 2x + e^{4x}$
 $h'(x) = 2 \cos 2x + 4e^{4x}$
 $h'(-0.9) = 2 \cos(-1.8) + 4e^{-3.6}$
 $= -0.345\dots < 0$
 $h'(-0.8) = 2 \cos(-1.6) + 4e^{-3.2}$
 $= 0.104\dots > 0$
 The change in sign of $h'(x)$ implies that the gradient changes from decreasing to increasing, so there is a turning point in the interval $-0.9 < x < -0.8$.

- b** Choose interval $[-0.8225, -0.8235]$ to test for root.
 $h'(-0.8235) = 2 \cos(-1.647) + 4e^{-3.294}$
 $= -0.00383\dots < 0$
 $h'(-0.8225) = 2 \cos(-1.645) + 4e^{-3.29}$
 $= 0.000744\dots > 0$
 There is a change of sign between -0.8225 and -0.8235 , so $-0.8225 < \alpha < -0.8235$, which gives $\alpha = -0.823$ correct to 3 d.p.

11 a



- b** The curves meet at one point, so there is one value of x that satisfies the equation $\sqrt{x} = \frac{2}{x}$. So $\sqrt{x} = \frac{2}{x}$ has **one** root.

- c** $f(x) = \sqrt{x} - \frac{2}{x}$
 $f(1) = \sqrt{1} - \frac{2}{1} = -1$
 $f(2) = \sqrt{2} - \frac{2}{2} = 0.414\dots$
 There is a change of sign, so there is a root, r , between $x = 1$ and $x = 2$.

- d** $\sqrt{x} = \frac{2}{x}$
 $x^{\frac{1}{2}} = \frac{2}{x}$
 $x^{\frac{1}{2}} \times x = 2$
 $x^{\frac{1}{2}+1} = 2$
 $x^{\frac{3}{2}} = 2$
 $\left(x^{\frac{3}{2}}\right)^2 = 2^2$
 $x^3 = 4$
 So $p = 3$ and $q = 4$.

- e** $x^{\frac{3}{2}} = 2$
 $\Rightarrow x = 2^{\frac{2}{3}} = \left(2^2\right)^{\frac{1}{3}} = 4^{\frac{1}{3}}$

- 12 a** $f(x) = x^4 - 21x - 18$
 $f(-0.9) = 0.6561 + 18.9 - 18 = 1.5561 > 0$
 $f(-0.8) = 0.4096 + 16.8 - 18 = -0.7904 < 0$
 The change of sign between -0.9 and -0.8 implies there is at least one root in the interval $[-0.9, -0.8]$.

12 b $f'(x) = 4x^3 - 21$

$$f'(x) = 0 \Rightarrow 4x^3 = 21$$

$$x = \sqrt[3]{\frac{21}{4}} = 1.738\dots$$

$$\begin{aligned} f(1.738) &= 1.738^4 - 21 \times 1.738 - 18 \\ &= -45.373\dots \end{aligned}$$

Stationary point is (1.74, -45.37) to 2 d.p.

c $f(x) = (x-3)(x^3 + ax^2 + bx + c)$

$$f(x) = x^4 + (a-3)x^3 + (b-3a)x^2 + (c-3b)x - 3c$$

Comparing coefficients...

$$a = 3, b = 9, c = 6$$

