Differentiation, Mixed Exercise 9

1 a $y = \ln x^2 = 2\ln x$

(using properties of logs)

$\therefore \frac{dy}{dx} = 2 \times \frac{1}{x} = \frac{2}{x}$

Alternative method:

When $y = \ln f(x)$, $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

(by the chain rule)

$\therefore y = \ln x^2 \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2} = \frac{2}{x}$

b $y = x^2 \sin 3x$

Using the product rule,

$\frac{dy}{dx} = x^2(3\cos 3x) + (\sin 3x) \times 2x$

$= 3x^2 \cos 3x + 2x \sin 3x$

2 a $2y = x - \sin x \cos x$

$\therefore y = \frac{x}{2} - \frac{1}{2}\sin x \cos x$

Using the product rule,

$\frac{dy}{dx} = \frac{1}{2} - \frac{1}{2}\left(\sin x(-\sin x) + \cos x \cos x\right)$

$= \frac{1}{2} + \frac{1}{2}\sin^2 x - \frac{1}{2}\cos^2 x$

$= \frac{1}{2}(1 - \cos^2 x) + \frac{1}{2}\sin^2 x$

$= \frac{1}{2}\sin^2 x + \frac{1}{2}\sin^2 x$

$= \sin^2 x$

b $y = \frac{x}{2} - \frac{1}{2}\sin x \cos x$

$\frac{dy}{dx} = \sin^2 x$

$\frac{d^2y}{dx^2} = 2\sin x \cos x = \sin 2x$

At points of inflection $\frac{d^2y}{dx^2} = 0$

i.e. $\sin 2x = 0$

$2x = \pi, 2\pi$ or 3π

$x = \frac{\pi}{2}, \pi$ or $\frac{3\pi}{2}$

When $x = \frac{\pi}{2}$, $y = \frac{\pi}{4}$

At $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} > 0$; at $x = \frac{3\pi}{4}$, $\frac{d^2y}{dx^2} < 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \frac{\pi}{2}$

When $x = \pi$, $y = \frac{\pi}{2}$

At $x = \frac{3\pi}{4}$, $\frac{d^2y}{dx^2} < 0$; at $x = \frac{5\pi}{4}$, $\frac{d^2y}{dx^2} > 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \pi$

When $x = \frac{3\pi}{2}$, $y = \frac{3\pi}{4}$

At $x = \frac{5\pi}{4}$, $\frac{d^2y}{dx^2} > 0$; at $x = \frac{7\pi}{4}$, $\frac{d^2y}{dx^2} < 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \frac{3\pi}{2}$

Hence the points of inflection are

$\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{3\pi}{4}\right)$

3 a $y = \dfrac{\sin x}{x}$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{x\cos x - \sin x \times 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

b $y = \ln\dfrac{1}{x^2+9} = \ln 1 - \ln(x^2+9) = -\ln(x^2+9)$

(by the laws of logarithms)

Using the chain rule:

$$\frac{dy}{dx} = -\frac{1}{x^2+9} \times 2x = -\frac{2x}{x^2+9}$$

4 a $f(x) = \dfrac{x}{x^2+2}$

$$f'(x) = \frac{(x^2+2)\times 1 - x \times 2x}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2}$$

The function is increasing when $f'(x) \geqslant 0$

i.e. $\dfrac{2-x^2}{(x^2+2)^2} \geqslant 0$

$x^2 \leqslant 2$

$-\sqrt{2} \leqslant x \leqslant \sqrt{2}$

Hence $f(x)$ is increasing on the interval $[-k, k]$ where $k = \sqrt{2}$.

b $f''(x) = \dfrac{-2x(x^2+2)^2 - 4x(x^2+2)(2-x^2)}{(x^2+2)^4}$

$$= \frac{2x(x^2+2)\left(-(x^2+2) - 2(2-x^2)\right)}{(x^2+2)^4} = \frac{2x(x^2+2)(x^2-6)}{(x^2+2)^4}$$

$f''(x)$ changes sign when the numerator $2x(x^2+2)(x^2-6)$ is zero

i.e. at $x = 0$ and $x = \pm\sqrt{6}$

where $y = 0$ and $y = \dfrac{\pm\sqrt{6}}{6+2}$

Points of inflection are

$(0, 0)$ and $\left(\pm\sqrt{6}, \pm\dfrac{\sqrt{6}}{8}\right)$

5 a $f(x) = 12\ln x + x^{\frac{3}{2}}, \quad x > 0$

$$f'(x) = \frac{12}{x} + \frac{3}{2}x^{\frac{1}{2}} = \frac{12}{x} + \frac{3}{2}\sqrt{x}$$

$f(x)$ is an increasing function when $f'(x) \geq 0$

As $x > 0$, $\dfrac{12}{x} + \dfrac{3}{2}\sqrt{x}$ is always positive.

$\therefore$ $f(x)$ is increasing for all $x > 0$.

b $f''(x) = -\dfrac{12}{x^2} + \dfrac{3}{4}x^{-\frac{1}{2}} = -\dfrac{12}{x^2} + \dfrac{3}{4\sqrt{x}}$

At a point of inflection $f''(x) = 0$

$$-\frac{12}{x^2} + \frac{3}{4\sqrt{x}} = 0$$

$$\frac{12}{x^2} = \frac{3}{4\sqrt{x}}$$

$$x^2 = 16\sqrt{x}$$

$$x^{\frac{3}{2}} = 16$$

$$x = \sqrt[3]{256}$$

$$f\left(\sqrt[3]{256}\right) = 12\ln(256)^{\frac{1}{3}} + 256^{\frac{1}{2}} = 4\ln 256 + 16 = 4\ln 2^8 + 16 = 32\ln 2 + 16$$

Coordinates of the point of inflection are

$\left(\sqrt[3]{256}, 32\ln 2 + 16\right)$

6 $y = \cos^2 x + \sin x$

$$\frac{dy}{dx} = -2\cos x \sin x + \cos x$$
$$= \cos x(1 - 2\sin x)$$

At stationary points $\frac{dy}{dx} = 0$

$\cos x(1 - 2\sin x) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

Solutions in the interval $(0, 2\pi]$ are

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ and $\frac{3\pi}{2}$

$x = \frac{\pi}{6} \Rightarrow y = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$x = \frac{\pi}{2} \Rightarrow y = 1$

$x = \frac{5\pi}{6} \Rightarrow y = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$x = \frac{3\pi}{2} \Rightarrow y = -1$

So the stationary points are

$\left(\frac{\pi}{6}, \frac{5}{4}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{5}{4}\right)$ and $\left(\frac{3\pi}{2}, -1\right)$

7 $y = x\sqrt{\sin x} = x(\sin x)^{\frac{1}{2}}$

$$\frac{dy}{dx} = x \times \frac{1}{2}(\sin x)^{-\frac{1}{2}} \cos x + (\sin x)^{\frac{1}{2}} \times 1$$
$$= \frac{1}{2}(\sin x)^{-\frac{1}{2}}(x\cos x + 2\sin x)$$

At the maximum point $\frac{dy}{dx} = 0$

$\frac{1}{2}(\sin x)^{-\frac{1}{2}}(x\cos x + 2\sin x) = 0$

$\therefore\ x\cos x + 2\sin x = 0$

(as $(\sin x)^{-\frac{1}{2}} = \frac{1}{\sqrt{\sin x}} \neq 0$)

Dividing through by $\cos x$ gives

$x + 2\tan x = 0$

So the x-coordinate of the maximum point satisfies $2\tan x + x = 0$.

8 a $f(x) = e^{0.5x} - x^2$

$f'(x) = 0.5e^{0.5x} - 2x$

b $f'(6) = -1.957... < 0$

$f'(7) = 2.557... > 0$

As the sign changes between $x = 6$ and $x = 7$ and $f'(x)$ is continuous, $f'(x) = 0$ has a root p between 6 and 7.

Therefore $y = f(x)$ has a stationary point at $x = p$ where $6 < p < 7$.

9 a $f(x) = e^{2x}\sin 2x$

$$f'(x) = e^{2x}(2\cos 2x) + \sin 2x(2e^{2x})$$
$$= 2e^{2x}(\cos 2x + \sin 2x)$$

At turning points $f'(x) = 0$

$2e^{2x}(\cos 2x + \sin 2x) = 0$

$\cos 2x + \sin 2x = 0$

$\sin 2x = -\cos 2x$

Divide both sides by $\cos 2x$:

$\tan 2x = -1$

$2x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

$\therefore x = \frac{3\pi}{8}$ or $\frac{7\pi}{8}$ (in the interval $0 < x < \pi$)

When $x = \frac{3\pi}{8}$, $y = \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}$

When $x = \frac{7\pi}{8}$, $y = -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}$

So the coordinates of the turning points are $\left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right)$ and $\left(\frac{7\pi}{8}, -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right)$.

9 b $f'(x) = 2e^{2x}(\cos 2x + \sin 2x)$

$$f''(x) = 2e^{2x}(-2\sin 2x + 2\cos 2x) + 4e^{2x}(\cos 2x + \sin 2x)$$
$$= e^{2x}(-4\sin 2x + 4\cos 2x + 4\cos 2x + 4\sin 2x)$$
$$= 8e^{2x}\cos 2x$$

c $f''\left(\frac{3\pi}{8}\right) = 8e^{\frac{3\pi}{4}}\cos\frac{3\pi}{4}$

$$= 8e^{\frac{3\pi}{4}}\left(-\frac{\sqrt{2}}{2}\right) = -4\sqrt{2}\,e^{\frac{3\pi}{4}} < 0$$

$\therefore \left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right)$ is a maximum.

$f''\left(\frac{7\pi}{8}\right) = 8e^{\frac{7\pi}{4}}\cos\frac{7\pi}{4}$

$$= 8e^{\frac{7\pi}{4}}\left(\frac{\sqrt{2}}{2}\right) = 4\sqrt{2}\,e^{\frac{7\pi}{4}} > 0$$

$\therefore \left(\frac{7\pi}{8}, -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right)$ is a minimum.

d At points of inflection $f''(x) = 0$

$8e^{2x}\cos 2x = 0$

$\cos 2x = 0$

$2x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\therefore x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

When $x = \frac{\pi}{4}$, $y = e^{\frac{\pi}{2}}\sin\frac{\pi}{2} = e^{\frac{\pi}{2}}$

When $x = \frac{3\pi}{4}$, $y = e^{\frac{3\pi}{2}}\sin\frac{3\pi}{2} = -e^{\frac{3\pi}{2}}$

Points of inflection are $\left(\frac{\pi}{4}, e^{\frac{\pi}{2}}\right)$ and $\left(\frac{3\pi}{4}, -e^{\frac{3\pi}{2}}\right)$.

10 $y = 2e^x + 3x^2 + 2$

$\frac{dy}{dx} = 2e^x + 6x$

When $x = 0$, $y = 4$ and $\frac{dy}{dx} = 2$

Equation of normal at (0, 4) is

$y - 4 = -\frac{1}{2}(x - 0)$

$2y - 8 = -x$

or $x + 2y - 8 = 0$

11 a $f(x) = 3\ln x + \frac{1}{x}$

$f'(x) = \frac{3}{x} - \frac{1}{x^2}$

At a stationary point $\frac{dy}{dx} = 0$

$\frac{3}{x} - \frac{1}{x^2} = 0$

$3x - 1 = 0$

$x = \frac{1}{3}$

So the x-coordinate of the stationary point P is $\frac{1}{3}$

b At the point Q, $x = 1$ so $y = f(1) = 1$

The gradient of the curve at point Q is $f'(1) = 3 - 1 = 2$

So the gradient of the normal to the curve at Q is $-\frac{1}{2}$

Equation of the normal at Q is

$y - 1 = -\frac{1}{2}(x - 1)$

i.e. $y = -\frac{1}{2}x + \frac{3}{2}$

12 a Let $f(x) = e^{2x}\cos x$

Then $f'(x) = e^{2x}(-\sin x) + \cos x(2e^{2x})$
$= e^{2x}(2\cos x - \sin x)$

Turning points occur when $f'(x) = 0$

$e^{2x}(2\cos x - \sin x) = 0$
$\sin x = 2\cos x$

Dividing both sides by $\cos x$ gives

$\tan x = 2$

b When $x = 0, y = f(0) = e^0 \cos 0 = 1$

The gradient of the curve at (0, 1) is
$f'(0) = e^0(2 - 0) = 2$

This is also the gradient of the tangent at (0, 1).

So the equation of the tangent at (0, 1) is

$y - 1 = 2(x - 0)$

$y = 2x + 1$

13 a $x = y^2 \ln y$

Using the product rule:

$$\frac{dx}{dy} = y^2\left(\frac{1}{y}\right) + \ln y \times 2y = y + 2y\ln y$$

b $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{y + 2y\ln y}$$

When $y = e$,

$$\frac{dy}{dx} = \frac{1}{e + 2e\ln e} = \frac{1}{3e}$$

14 a $f(x) = (x^3 - 2x)e^{-x}$
$f'(x) = (x^3 - 2x)(-e^{-x}) + (3x^2 - 2)e^{-x}$
$= e^{-x}(-x^3 + 3x^2 + 2x - 2)$

b When $x = 0$, $f'(x) = -2$

Gradient of normal is $\frac{1}{2}$

$\therefore$ equation of normal to the curve at the origin is

$y = \frac{1}{2}x$

This line will intersect the curve again when

$\frac{1}{2}x = (x^3 - 2x)e^{-x}$

$1 = 2(x^2 - 2)e^{-x}$

$e^x = 2x^2 - 4$

$2x^2 = e^x + 4$

15 a $f(x) = x(1 + x)\ln x = (x + x^2)\ln x$

$f'(x) = (x + x^2) \times \frac{1}{x} + \ln x \times (1 + 2x)$
$= 1 + x + (1 + 2x)\ln x$

b At minimum point $A, f'(x) = 0$
$1 + x + (1 + 2x)\ln x = 0$
$(1 + 2x)\ln x = -(1 + x)$
$\ln x = -\frac{1 + x}{1 + 2x}$
So x-coordinate of A is the solution to the equation $x = e^{-\frac{1+x}{1+2x}}$

16 a $x = 4t - 3,\ y = \frac{8}{t^2} = 8t^{-2}$

$\frac{dx}{dt} = 4,\ \frac{dy}{dt} = -16t^{-3}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-16t^{-3}}{4} = \frac{-4}{t^3}$$

16 b When $t = 2$, the curve has gradient $\frac{dy}{dx} = \frac{-4}{2^3} = -\frac{1}{2}$

$\therefore$ the normal has gradient 2.

Also, when $t = 2$, $x = 5$ and $y = 2$, so the point A has coordinates (5, 2).

$\therefore$ the equation of the normal at A is

$y - 2 = 2(x - 5)$

i.e. $y = 2x - 8$

17 $x = 2t,\ y = t^2$

$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t$$

At the point P where $t = 3$, the gradient of the curve is $\frac{dy}{dx} = 3$

$\therefore$ gradient of the normal is $-\frac{1}{3}$

Also, when $t = 3$, the coordinates are (6, 9).

$\therefore$ the equation of the normal at P is

$y - 9 = -\frac{1}{3}(x - 6)$

i.e. $3y + x = 33$

18 $x = t^3,\ y = t^2$

$\frac{dx}{dt} = 3t^2,\ \frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2} = \frac{2}{3t}$$

At the point (1, 1) the value of t is 1.

$\therefore$ the gradient of the curve is $\frac{2}{3}$, which is also the gradient of the tangent.

$\therefore$ the equation of the tangent is

$y - 1 = \frac{2}{3}(x - 1)$

i.e. $y = \frac{2}{3}x + \frac{1}{3}$

19 a $x = 2\cos t + \sin 2t,\ y = \cos t - 2\sin 2t$

$\frac{dx}{dt} = -2\sin t + 2\cos 2t$

$\frac{dy}{dt} = -\sin t - 4\cos 2t$

b $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t - 4\cos 2t}{-2\sin t + 2\cos 2t}$$

When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{-\frac{1}{\sqrt{2}} - 0}{-\frac{2}{\sqrt{2}} + 0} = \frac{1}{2}$

19 c The gradient of the normal at the point P where $t = \frac{\pi}{4}$ is –2.

The coordinates of P are found by substituting $t = \frac{\pi}{4}$ into the parametric equations:

$$x = \frac{2}{\sqrt{2}} + 1,\ y = \frac{1}{\sqrt{2}} - 2$$

$\therefore$ the equation of the normal at P is

$$y - \left(\frac{1}{\sqrt{2}} - 2\right) = -2\left(x - \left(\frac{2}{\sqrt{2}} + 1\right)\right)$$

$$y - \frac{1}{\sqrt{2}} + 2 = -2x + 2\sqrt{2} + 2$$

i.e. $y + 2x = \frac{5\sqrt{2}}{2}$

20 a $x = 2t + 3,\ y = t^3 - 4t$

At point A, where $t = -1$, $x = 1$ and $y = 3$.

$\therefore$ the coordinates of A are (1, 3).

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 3t^2 - 4$$

$$\therefore \frac{dy}{dx} = \frac{3t^2 - 4}{2}$$

At the point A, $\frac{dy}{dx} = -\frac{1}{2}$

$\therefore$ gradient of the tangent at A is $-\frac{1}{2}$

Equation of the tangent at A is

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$2y - 6 = -x + 1$$

i.e. $2y + x = 7$

b The tangent line l meets the curve C at points A and B.

Substitute $x = 2t + 3$ and $y = t^3 - 4t$ into the equation of l:

$$2(t^3 - 4t) + (2t + 3) = 7$$

$$2t^3 - 6t = 4$$

$$t^3 - 3t - 2 = 0$$

At point A, $t = -1$, so $t = -1$ is a root of this equation, and hence $(t + 1)$ is a factor of the left-hand side expression.

$$\begin{aligned} t^3 - 3t - 2 &= (t+1)(t^2 - t - 2) \\ &= (t+1)(t+1)(t-2) \\ &= 2(t+1)^2(t-2) \end{aligned}$$

So line l meets the curve C at $t = -1$ (repeated root because the line is tangent to the curve there) and at $t = 2$.

Therefore, at point B, $t = 2$.

21 The rate of change of V is $\frac{dV}{dt}$

$$\therefore \frac{dV}{dt} \propto V$$

i.e. $\frac{dV}{dt} = -kV$

where k is a positive constant.
(The negative sign is needed as the value of the car is *decreasing*.)

22 The rate of change of mass is $\frac{dM}{dt}$

$$\therefore \frac{dM}{dt} \propto M$$

i.e. $\frac{dM}{dt} = -kM$

where k is a positive constant.
(The negative sign represents *loss* of mass.)

23 The rate of change of pondweed is $\frac{dP}{dt}$

The growth rate is proportional to P:

growth rate $\propto P$

i.e. growth rate $= kP$
where k is a positive constant.

But pondweed is also being removed at a constant rate Q.

$$\therefore \frac{dP}{dt} = \text{growth rate} - \text{removal rate}$$
$$\frac{dP}{dt} = kP - Q$$

24 The rate of increase of the radius is $\frac{dr}{dt}$

$\therefore \frac{dr}{dt} \propto \frac{1}{r}$, as the rate is *inversely* proportional to the radius.

Hence $\frac{dr}{dt} = \frac{k}{r}$
where k is the constant of proportion.

25 The rate of change of temperature is $\frac{d\theta}{dt}$

$$\therefore \frac{d\theta}{dt} \propto (\theta - \theta_0)$$

i.e. $\frac{d\theta}{dt} = -k(\theta - \theta_0)$,
where k is a positive constant.
(The negative sign indicates that the temperature is decreasing, i.e. *loss* of temperature.)

26 a $x = 4\cos 2t,\ y = 3\sin t$

The point $A\left(2, \frac{3}{2}\right)$ is on the curve, so

$$4\cos 2t = 2 \text{ and } 3\sin t = \frac{3}{2}$$

$$\cos 2t = \frac{1}{2} \text{ and } \sin t = \frac{1}{2}$$

The only value of t in the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$ that satisfies both equations is $\frac{\pi}{6}$. Therefore $t = \frac{\pi}{6}$ at the point A.

b $\frac{dx}{dt} = -8\sin 2t,\ \frac{dy}{dt} = 3\cos t$

$$\therefore \frac{dy}{dx} = \frac{3\cos t}{-8\sin 2t}$$
$$= -\frac{3\cos t}{16\sin t\cos t}$$
(using a double angle formula)
$$= -\frac{3}{16\sin t}$$
$$= -\frac{3}{16}\operatorname{cosec} t$$

c At point A, where $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{3}{8}$

$\therefore$ gradient of the normal at A is $\frac{8}{3}$

Equation of the normal is

$$y - \frac{3}{2} = \frac{8}{3}(x - 2)$$

Multiply through by 6 and rearrange to give:

$$6y - 9 = 16x - 32$$
$$6y - 16x + 23 = 0$$

26 d To find where the normal cuts the curve, substitute $x = 4\cos 2t$ and $y = 3\sin t$ into the equation of the normal:

$$6(3\sin t) - 16(4\cos 2t) + 23 = 0$$
$$18\sin t - 64\cos 2t + 23 = 0$$
$$18\sin t - 64(1 - 2\sin^2 t) + 23 = 0$$
(using a double angle formula)
$$128\sin^2 t + 18\sin t - 41 = 0$$

But $\sin t = \frac{1}{2}$ is one solution of this equation, as point A lies on the line and on the curve. So

$$128\sin^2 t + 18\sin t - 41$$
$$= (2\sin t - 1)(64\sin t + 41)$$

$$\therefore (2\sin t - 1)(64\sin t + 41) = 0$$

Therefore, at point B, $\sin t = -\frac{41}{64}$

$\therefore$ the y-coordinate of point B is
$3 \times \left(-\frac{41}{64}\right) = -\frac{123}{64}$

27 a $x = a\sin^2 t,\ y = a\cos t$

$$\frac{dx}{dt} = 2a\sin t\cos t,\ \frac{dy}{dt} = -a\sin t$$

$$\therefore \frac{dy}{dx} = \frac{-a\sin t}{2a\sin t\cos t} = \frac{-1}{2\cos t} = -\frac{1}{2}\sec t$$

b As $P\left(\frac{3}{4}a, \frac{1}{2}a\right)$ lies on the curve,

$$a\sin^2 t = \frac{3}{4}a \text{ and } a\cos t = \frac{1}{2}a$$
$$\sin t = \pm\frac{\sqrt{3}}{2} \text{ and } \cos t = \frac{1}{2}$$

The only value of t in the interval

$0 \leqslant t \leqslant \frac{\pi}{2}$ that satisfies both equations

is $\frac{\pi}{3}$. Therefore $t = \frac{\pi}{3}$ at the point P.

Gradient of the curve at point P is
$-\frac{1}{2}\sec\frac{\pi}{3} = -1.$

$\therefore$ equation of the tangent at P is

$$y - \frac{1}{2}a = -1\left(x - \frac{3}{4}a\right)$$
$$y - \frac{1}{2}a = -x + \frac{3}{4}a$$

Multiply equation by 4 and rearrange to give
$$4y + 4x = 5a$$

c Equation of the tangent at C is
$4y + 4x = 5a$

At A, $x = 0 \Rightarrow y = \frac{5a}{4}$

At B, $y = 0 \Rightarrow x = \frac{5a}{4}$

Area of $AOB = \frac{1}{2}\left(\frac{5a}{4}\right)^2 = \frac{25}{32}a^2$,

which is of the form ka^2 with $k = \frac{25}{32}$

28 $x = (t+1)^2,\ y = \frac{1}{2}t^3 + 3$

$$\frac{dx}{dt} = 2(t+1),\ \frac{dy}{dt} = \frac{3}{2}t^2$$

$$\therefore \frac{dy}{dx} = \frac{\frac{3}{2}t^2}{2(t+1)} = \frac{3t^2}{4(t+1)}$$

When $t = 2$, $\frac{dy}{dx} = \frac{3\times 4}{4\times 3} = 1$

$\therefore$ gradient of the normal at the point P, where $t = 2$, is -1.

The coordinates of P are (9, 7).

$\therefore$ equation of the normal is

$$y - 7 = -1(x - 9)$$
$$y - 7 = -x + 9$$
i.e. $y + x = 16$

29 $5x^2 + 5y^2 - 6xy = 13$

Differentiate with respect to x:

$$10x + 10y\frac{dy}{dx} - 6\left(x\frac{dy}{dx} + y\right) = 0$$

$$(10y - 6x)\frac{dy}{dx} = 6y - 10x$$
$$\therefore \frac{dy}{dx} = \frac{6y - 10x}{10y - 6x}$$

At the point (1, 2)

$$\frac{dy}{dx} = \frac{12 - 10}{20 - 6} = \frac{2}{14} = \frac{1}{7}$$

So the gradient of the curve at (1, 2) is $\frac{1}{7}$

30 $e^{2x} + e^{2y} = xy$

Differentiate with respect to x:

$$2e^{2x} + 2e^{2y}\frac{dy}{dx} = x\frac{dy}{dx} + y \times 1$$

$$2e^{2y}\frac{dy}{dx} - x\frac{dy}{dx} = y - 2e^{2x}$$

$$(2e^{2y} - x)\frac{dy}{dx} = y - 2e^{2x}$$

$$\therefore \frac{dy}{dx} = \frac{y - 2e^{2x}}{2e^{2y} - x}$$

31 $y^3 + 3xy^2 - x^3 = 3$

Differentiate with respect to x:

$$3y^2\frac{dy}{dx} + \left(3x \times 2y\frac{dy}{dx} + y^2 \times 3\right) - 3x^2 = 0$$

$$(3y^2 + 6xy)\frac{dy}{dx} = 3x^2 - 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{3(x^2 - y^2)}{3y(y + 2x)} = \frac{x^2 - y^2}{y(y + 2x)}$$

Turning points occur when $\frac{dy}{dx} = 0$

$$\frac{x^2 - y^2}{y(y + 2x)} = 0$$
$$x^2 = y^2$$
$$x = \pm y$$

When $x = y$, $y^3 + 3y^3 - y^3 = 3$
so $3y^3 = 3$
$y = 1$ and hence $x = 1$

When $x = -y$, $y^3 - 3y^3 + y^3 = 3$
so $-y^3 = 3$
$y = \sqrt[3]{-3}$ and hence $x = -\sqrt[3]{-3}$

$\therefore$ the coordinates of the turning points are (1, 1) and $(-\sqrt[3]{-3}, \sqrt[3]{-3})$.

32 a $(1+x)(2+y)=x^2+y^2$

Differentiate with respect to x:

$$(1+x)\left(\frac{dy}{dx}\right)+(2+y)(1)=2x+2y\frac{dy}{dx}$$

$$(1+x-2y)\frac{dy}{dx}=2x-y-2$$

$$\therefore \frac{dy}{dx}=\frac{2x-y-2}{1+x-2y}$$

b When the curve meets the y-axis, $x = 0$.

Substitute $x = 0$ into the equation of the curve:

$2+y=y^2$

i.e. $y^2-y-2=0$

$(y-2)(y+1)=0$

$y=2$ or $y=-1$

At $(0, 2)$, $\frac{dy}{dx}=\frac{0-2-2}{1+0-4}=\frac{4}{3}$

At $(0, -1)$, $\frac{dy}{dx}=\frac{0+1-2}{1+0+2}=-\frac{1}{3}$

c A tangent that is parallel to the y-axis has infinite gradient.

For $\frac{dy}{dx}=\frac{2x-y-2}{1+x-2y}$ to be infinite,

the denominator $1+x-2y=0$,

i.e. $x=2y-1$

Substitute $x=2y-1$ into the equation of the curve:

$(1+2y-1)(2+y)=(2y-1)^2+y^2$

$2y^2+4y=4y^2-4y+1+y^2$

$3y^2-8y+1=0$

$y=\frac{8\pm\sqrt{64-12}}{6}=\frac{4\pm\sqrt{13}}{3}$

When $y-\frac{4+\sqrt{13}}{3}$, $x=\frac{5+2\sqrt{13}}{3}$

When $y=\frac{4-\sqrt{13}}{3}$, $x=\frac{5-2\sqrt{13}}{3}$

$\therefore$ there are two points at which the tangents are parallel to the y-axis.

They are $\left(\frac{5+2\sqrt{13}}{3}, \frac{4+\sqrt{13}}{3}\right)$ and $\left(\frac{5-2\sqrt{13}}{3}, \frac{4-\sqrt{13}}{3}\right)$.

33 $7x^2 + 48xy - 7y^2 + 75 = 0$

Implicit differentiation with respect to x gives

$$14x + 48\left(x\frac{dy}{dx} + y\right) - 14y\frac{dy}{dx} = 0$$
$$(48x - 14y)\frac{dy}{dx} = -14x - 48y$$
$$\therefore \frac{dy}{dx} = \frac{-14x - 48y}{48x - 14y} = \frac{7x + 24y}{7y - 24x}$$

When $\frac{dy}{dx} = \frac{2}{11}$,

$$\frac{7x + 24y}{7y - 24x} = \frac{2}{11}$$
$$14y - 48x = 77x + 264y$$
$$125x + 250y = 0$$
$$\therefore x + 2y = 0$$

So the coordinates of the points at which the gradient is $\frac{2}{11}$ satisfy $x + 2y = 0$, which means that the points lie on the line $x + 2y = 0$.

34 $y = x^x$

Take natural logs of both sides:

$$\ln y = \ln x^x$$

$$\ln y = x \ln x \quad \text{(using properties of logarithms)}$$

Differentiate with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln x \times 1$$
$$= 1 + \ln x$$

$$\therefore \frac{dy}{dx} = y(1 + \ln x)$$

But $y = x^x$

$$\therefore \frac{dy}{dx} = x^x(1 + \ln x)$$

35 a $a^x = e^{kx}$

Take natural logs of both sides:

$$\ln a^x = \ln e^{kx}$$
$$x \ln a = kx$$

As this is true for all values of x, $k = \ln a$.

b Taking $a = 2$,

$$y = 2^x = e^{kx} \text{ where } k = \ln 2$$

$$\frac{dy}{dx} = ke^{kx} = (\ln 2)e^{(\ln 2)x} = 2^x \ln 2$$

c At the point (2, 4), $x = 2$.

$\therefore$ gradient of the curve at (2, 4) is

$$\frac{dy}{dx} = 2^2 \ln 2 = 4 \ln 2 = \ln 2^4 = \ln 16$$

36 a $P = P_0(1.09)^t$

Take natural logs of both sides:

$$\ln P = \ln\left(P_0(1.09)^t\right)$$
$$= \ln P_0 + \ln(1.09)^t$$
$$= \ln P_0 + t \ln 1.09$$

$$\therefore t \ln 1.09 = \ln P - \ln P_0$$

$$t = \frac{\ln P - \ln P_0}{\ln 1.09} \quad \text{or} \quad \frac{\ln\left(\frac{P}{P_0}\right)}{\ln 1.09}$$

b When $t = T$, $P = 2P_0$.

Substituting these into the expression in part **a** gives

$$T = \frac{\ln 2}{\ln 1.09} = 8.04 \text{ (3 s.f.)}$$

36 c $\frac{dP}{dt} = P_0(1.09)^t(\ln 1.09)$

When $t = T$, $P = 2P_0$ so $(1.09)^T = 2$

Hence $\frac{dP}{dt} = P_0(1.09)^T(\ln 1.09)$
$= P_0 \times 2 \times \ln 1.09$
$= 0.172P_0$ (3 s.f.)

37 a $y = \ln(\sin x)$

$\frac{dy}{dx} = \cos x \times \frac{1}{\sin x} = \cot x$

At a stationary point $\frac{dy}{dx} = 0$

$\cot x = 0 \Rightarrow x = \frac{\pi}{2}$

(in the interval $0 < x < \pi$)

When $x = \frac{\pi}{2}$, $y = \ln\left(\sin\frac{\pi}{2}\right) = \ln 1 = 0$

$\therefore$ stationary point is at $\left(\frac{\pi}{2}, 0\right)$.

b $\frac{d^2y}{dx^2} = -\text{cosec}^2 x$

$\text{cosec}^2 x = \frac{1}{\sin^2 x} > 0$ for all $0 < x < \pi$

$\therefore \frac{d^2y}{dx^2} < 0$ for all $0 < x < \pi$

Hence the curve C is concave for all values of x in its domain.

38 a $m = 40\text{e}^{-0.244t}$

After 9 months, $t = 0.75$, so
$m = 40\text{e}^{-0.244\times0.75} = 40\text{e}^{-0.183} = 33.31...$

b $\frac{dm}{dt} = -0.244 \times 40\text{e}^{-0.244t} = -9.76\text{e}^{-0.244t}$

c The negative sign indicates that the mass is decreasing.

39 a $f(x) = \frac{\cos 2x}{\text{e}^x}$

$f'(x) = \frac{-2\text{e}^x \sin 2x - \text{e}^x \cos 2x}{\text{e}^{2x}}$

$= -\frac{2\sin 2x + \cos 2x}{\text{e}^x}$

At A and B, $f'(x) = 0$
$2\sin 2x + \cos 2x = 0$
$2\tan 2x + 1 = 0$
$\tan 2x = -0.5$
$2x = 2.678$ or 5.820
$x = 1.339$ or 2.910
(in the interval $0 \leqslant x \leqslant \pi$)
$x = 1.339 \Rightarrow y = f(x) = -0.2344$

$x = 2.910 \Rightarrow y = f(x) = 0.04874$

Therefore, to 3 significant figures:
coordinates of A are (1.34, −0.234);
coordinates of B are (2.91, 0.0487).

b The curve of $y = 2 + 4f(x-4)$ is a transformation of $f(x)$, obtained via a translation of 4 units to the right, a stretch by a factor of 4 in the y-direction, and then a translation of 2 units upwards.

Turning points are:
minimum $(1.34 + 4, -0.234 \times 4 + 2)$ and
maximum $(2.91 + 4, 0.0487 \times 4 + 2)$,
i.e. minimum $(5.34, 1.06)$
and maximum $(6.91, 2.19)$.

c $f''(x)$

$= -\frac{\text{e}^x(4\cos 2x - 2\sin 2x) - \text{e}^x(2\sin 2x + \cos 2x)}{\text{e}^{2x}}$

$= \frac{4\sin 2x - 3\cos 2x}{\text{e}^x}$

f(x) is concave when $f''(x) \leqslant 0$
$f''(x) = 0$ when
$4\sin 2x - 3\cos 2x = 0$

$\tan 2x = \frac{3}{4}$

$2x = 0.644$ or 3.785

$x = 0.322$ or 1.893
The curve has a minimum point and hence is convex between these values, so it is concave for
$0 \leqslant x \leqslant 0.322$ and $1.892 \leqslant x \leqslant \pi$.

Challenge

a $y = 2\sin 2t,\ x = 5\cos\left(t + \frac{\pi}{12}\right)$

$\frac{dy}{dt} = 4\cos 2t,\ \frac{dx}{dt} = -5\sin\left(t + \frac{\pi}{12}\right)$

$\therefore \frac{dy}{dx} = -\frac{4\cos 2t}{5\sin\left(t + \frac{\pi}{12}\right)}$

b $\frac{dy}{dx} = 0$ when $4\cos 2t = 0$

$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ or $\frac{7\pi}{2}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

(in the interval $0 \leqslant x \leqslant 2\pi$)

$t = \frac{\pi}{4} \Rightarrow x = 5\cos\left(\frac{\pi}{3}\right) = \frac{5}{2}$

and $y = 2\sin\frac{\pi}{2} = 2$, i.e. point $\left(\frac{5}{2}, 2\right)$

$t = \frac{3\pi}{4} \Rightarrow x = 5\cos\left(\frac{5\pi}{6}\right) = -\frac{5\sqrt{3}}{2}$

and $y = 2\sin\frac{3\pi}{2} = -2$, i.e. point $\left(-\frac{5\sqrt{3}}{2}, -2\right)$

$t = \frac{5\pi}{4} \Rightarrow x = 5\cos\left(\frac{4\pi}{3}\right) = -\frac{5}{2}$

and $y = 2\sin\frac{5\pi}{2} = 2$, i.e. point $\left(-\frac{5}{2}, 2\right)$

$t = \frac{7\pi}{4} \Rightarrow x = 5\cos\left(\frac{11\pi}{6}\right) = \frac{5\sqrt{3}}{2}$

and $y = 2\sin\frac{7\pi}{2} = -2$, i.e. point $\left(\frac{5\sqrt{3}}{2}, -2\right)$

c The curve cuts the x-axis when $y = 0$,
i.e. when $2\sin 2t = 0$

$2t = 0, \pi, 2\pi, 3\pi, 4\pi$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$t = 0 \Rightarrow x = 5\cos\frac{\pi}{12} = 4.83$, i.e. $(4.83, 0)$

with gradient $\frac{dy}{dx} = \frac{-4}{5\sin\frac{\pi}{12}} = -3.09$

$t = \frac{\pi}{2} \Rightarrow x = 5\cos\frac{7\pi}{12} = -1.29$, i.e. $(-1.29, 0)$

with gradient $\frac{dy}{dx} = \frac{4}{5\sin\frac{7\pi}{12}} = 0.828$

$t = \pi \Rightarrow x = 5\cos\frac{13\pi}{12} = -4.83$, i.e. $(-4.83, 0)$

with gradient $\frac{dy}{dx} = \frac{-4}{5\sin\frac{13\pi}{12}} = 3.09$

$t = \frac{3\pi}{2} \Rightarrow x = 5\cos\frac{19\pi}{12} = 1.29$, i.e. $(1.29, 0)$

with gradient $\frac{dy}{dx} = \frac{4}{5\sin\frac{19\pi}{12}} = -0.828$

The curve cuts the y-axis when $x = 0$.

i.e. when $5\cos\left(t + \frac{\pi}{12}\right) = 0$

$t + \frac{\pi}{12} = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \frac{5\pi}{12}, \frac{17\pi}{12}$

$t = \frac{5\pi}{12} \Rightarrow y = 2\sin\frac{5\pi}{6} = 1$, i.e. $(0, 1)$

with gradient $\frac{dy}{dx} = \frac{-4\cos\frac{5\pi}{6}}{5\sin\frac{\pi}{2}} = 0.693$

$t = \frac{17\pi}{12} \Rightarrow y = 2\sin\frac{17\pi}{6} = 1$, i.e. $(0, 1)$

with gradient $\frac{dy}{dx} = \frac{-4\cos\frac{17\pi}{6}}{5\sin\frac{3\pi}{2}} = -0.693$

So the curve cuts the y-axis twice at (0, 1) with gradients 0.693 and −0.693.

d $\frac{dx}{dy} = -\frac{5\sin\left(t + \frac{\pi}{12}\right)}{4\cos 2t}$

$\frac{dx}{dy} = 0$ when $\sin\left(t + \frac{\pi}{12}\right) = 0$

$t + \frac{\pi}{12} = \pi, 2\pi$

$t = \frac{11\pi}{12}, \frac{23\pi}{12}$

$t = \frac{11\pi}{12} \Rightarrow y = 2\sin\frac{11\pi}{6} = -1$

and $x = 5\cos\left(\frac{11\pi}{12} + \frac{\pi}{12}\right) = -5$

$t = \frac{23\pi}{12} \Rightarrow y = 2\sin\frac{23\pi}{6} = -1$

and $x = 5\cos\left(\frac{23\pi}{12} + \frac{\pi}{12}\right) = 5$

So points where curve is vertical are
(−5, −1) and (5, −1).

e