Differentiation 9I

1 a $f(x) = x^3 - 3x^2 + x - 2$
$f'(x) = 3x^2 - 6x + 1$
$f''(x) = 6x - 6$

i $f(x)$ is convex when $f''(x) \geqslant 0$
$6x - 6 \geqslant 0$ for $x \geqslant 1$
So $f(x)$ is convex for all $x \geqslant 1$
or on the interval $[1, \infty)$.

ii $f(x)$ is concave when $f''(x) \leqslant 0$
$6x - 6 \leqslant 0$ for $x \leqslant 1$
So $f(x)$ is concave for all $x \leqslant 1$
or on the interval $(-\infty, 1]$.

b $f(x) = x^4 - 3x^3 + 2x - 1$
$f'(x) = 4x^3 - 9x^2 + 2$
$f''(x) = 12x^2 - 18x = 6x(2x - 3)$

i $f(x)$ is convex when $f''(x) \geqslant 0$
$6x(2x - 3) \geqslant 0$ for $x \leqslant 0$ or $x \geqslant \frac{3}{2}$
So $f(x)$ is convex for $x \leqslant 0$ or $x \geqslant \frac{3}{2}$,
or on $(-\infty, 0] \cup \left[\frac{3}{2}, \infty\right)$.

ii $f(x)$ is concave when $f''(x) \leqslant 0$
$6x(2x - 3) \leqslant 0$ for $0 \leqslant x \leqslant \frac{3}{2}$
So $f(x)$ is concave for all $0 \leqslant x \leqslant \frac{3}{2}$
or on the interval $\left[0, \frac{3}{2}\right]$.

c $f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$

i $f(x)$ is convex when $f''(x) \geqslant 0$
$-\sin x \geqslant 0$ for $\pi \leqslant x \leqslant 2\pi$
So $f(x)$ is convex on the interval
$[\pi, 2\pi]$.

ii $f(x)$ is concave when $f''(x) \leqslant 0$
$-\sin x \leqslant 0$ for $\leqslant 0\ x \leqslant \pi$
So $f(x)$ is concave on the interval
$[0, \pi]$.

d $f(x) = -x^2 + 3x - 7$
$f'(x) = -2x + 3$
$f''(x) = -2$

i $f(x)$ is convex when $f''(x) \geqslant 0$
So $f(x)$ is not convex anywhere.

ii $f(x)$ is concave when $f''(x) \leqslant 0$
So $f(x)$ is concave for all $x \in \mathbb{R}$
or on the interval $(-\infty, \infty)$.

e $f(x) = e^x - x^2$
$f'(x) = e^x - 2x$
$f''(x) = e^x - 2$

i $f(x)$ is convex when $f''(x) \geqslant 0$
$e^x - 2 \geqslant 0$ for $x \geqslant \ln 2$
So $f(x)$ is convex on $[\ln 2, \infty)$.

ii $f(x)$ is concave when $f''(x) \leqslant 0$
$e^x - 2 \leqslant 0$ for $x \leqslant \ln 2$
So $f(x)$ is concave on $(-\infty, \ln 2]$.

f $f(x) = \ln x, \quad x > 0$
$f'(x) = \frac{1}{x}$
$f''(x) = -\frac{1}{x^2}$

i $f(x)$ is convex when $f''(x) \geqslant 0$
But $-\frac{1}{x^2} < 0$ for all $x > 0$
So $f(x)$ is not convex anywhere.

ii $f(x)$ is concave when $f''(x) \leqslant 0$
$-\frac{1}{x^2} < 0$ for all $x > 0$
So $f(x)$ is concave on $(0, \infty)$.

2 a Let $y = \mathrm{f}(x)$. Then $x = \sin y$.

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

so $\mathrm{f}'(x) = \dfrac{1}{\sqrt{1-x^2}}$

b $\mathrm{f}(x) = \arcsin x$

$$\mathrm{f}'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$\mathrm{f}''(x) = (-2x)\left(-\frac{1}{2}\right)(1-x^2)^{-\frac{3}{2}} = \frac{x}{(1-x^2)^{\frac{3}{2}}}$$

On the interval $(-1, 0)$, $x < 0$

$\therefore\ \mathrm{f}''(x) \leqslant 0$

So $\mathrm{f}(x)$ is concave on the interval $(-1, 0)$.

c On the interval $(0, 1)$, $x > 0$

$\therefore\ \mathrm{f}''(x) \geqslant 0$

So $\mathrm{f}(x)$ is convex on the interval $(0, 1)$.

d $\mathrm{f}(x)$ changes from concave to convex at $x = 0$

When $x = 0$, $y = 0$.

$\therefore$ point of inflection is $(0, 0)$.

3 a $\mathrm{f}(x) = \cos^2 x - 2\sin x$

$$\mathrm{f}'(x) = -2\cos x \sin x - 2\cos x$$

$$\begin{aligned}\mathrm{f}''(x) &= -2(\cos^2 x - \sin^2 x) + 2\sin x \\ &= -2(1 - 2\sin^2 x) + 2\sin x \\ &= -2 + 4\sin^2 x + 2\sin x \\ &= 2(2\sin^2 x + \sin x - 1)\end{aligned}$$

At points of inflection $\mathrm{f}''(x) = 0$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$$\sin x = \frac{1}{2} \text{ or } -1$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}$$

Check the sign of $\mathrm{f}''(x)$ on either side of each point:

$$\mathrm{f}''(0) = 2(0+0-1) < 0$$

$$\mathrm{f}''\left(\frac{\pi}{2}\right) = 2(2+1-1) > 0$$

$\Rightarrow x = \dfrac{\pi}{6}$ is an inflection point

$$\mathrm{f}''(\pi) = 2(0+0-1) < 0$$

$\Rightarrow x = \dfrac{5\pi}{6}$ is an inflection point

$$\mathrm{f}''(2\pi) = 2(0+0-1) < 0$$

$\Rightarrow x = \dfrac{3\pi}{2}$ is not an inflection point

$$x = \frac{\pi}{6} \Rightarrow y = \left(\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{1}{2}\right) = -\frac{1}{4}$$

$$x = \frac{5\pi}{6} \Rightarrow y = \left(-\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{1}{2}\right) = -\frac{1}{4}$$

So the points of inflection are $\left(\dfrac{\pi}{6}, -\dfrac{1}{4}\right)$ and $\left(\dfrac{5\pi}{6}, -\dfrac{1}{4}\right)$.

3 b $f(x) = -\frac{x^3 - 2x^2 + x - 1}{x - 2}$

$= -\left(x^2 + 1 + \frac{1}{x-2}\right)$

$f'(x) = \frac{1}{(x-2)^2} - 2x$

$f''(x) = -\frac{2}{(x-2)^3} - 2 = -2\left(\frac{1}{(x-2)^3} + 1\right)$

At points of inflection $f''(x) = 0$

$\frac{1}{(x-2)^3} + 1 = 0$

$(x-2)^3 = -1$

$x - 2 = -1 \quad \therefore \; x = 1$

Check the sign of $f''(x)$ on either side of $x = 1$:

$f''(0.5) = -1.407... < 0$

$f''(1.5) = 14 > 0$

$\therefore \; x = 1$ is a point of inflection

When $x = 1$, $y = -(1 + 1 - 1) = -1$

So the point of inflection is $(1, -1)$.

c $f(x) = \frac{x^3}{x^2 - 4} = x + \frac{2}{x-2} + \frac{2}{x+2}$

$f'(x) = 1 - \frac{2}{(x-2)^2} - \frac{2}{(x+2)^2}$

$f''(x) = \frac{4}{(x-2)^3} + \frac{4}{(x+2)^3}$

$= 4\left(\frac{1}{(x-2)^3} + \frac{1}{(x+2)^3}\right)$

At points of inflection $f''(x) = 0$

$\frac{1}{(x-2)^3} + \frac{1}{(x+2)^3} = 0$

$(x-2)^3 = -(x+2)^3$

$x - 2 = -(x+2)$

$\therefore \; x = 0$

Check the sign of $f''(x)$ on either side of $x = 0$:

$f''(-1) = \frac{104}{27} > 0$

$f''(1) = -\frac{104}{27} < 0$

$\therefore \; x = 0$ is a point of inflection

When $x = 0$, $y = 0$

So the point of inflection is $(0, 0)$.

4 $f(x) = 2x^2 \ln x$

$f'(x) = 2x^2\left(\frac{1}{x}\right) + 4x\ln x = 2x(1 + 2\ln x)$

$f''(x) = 2x\left(\frac{2}{x}\right) + 2(1 + 2\ln x) = 6 + 4\ln x$

At a point of inflection $f''(x) = 0$

$6 + 4\ln x = 0 \Rightarrow \ln x = -\frac{3}{2} \Rightarrow x = e^{-\frac{3}{2}}$

So there is one point of inflection, where $x = e^{-\frac{3}{2}}$

5 a $y = e^x(x^2 - 2x + 2)$

$\frac{dy}{dx} = e^x(2x - 2) + e^x(x^2 - 2x + 2) = e^x x^2$

At stationary points $\frac{dy}{dx} = 0$

$e^x x^2 = 0$ when $x = 0$ and $y = 2$

$\therefore$ stationary point at (0, 2)

$\frac{d^2y}{dx^2} = 2xe^x + e^x x^2 = e^x x(x + 2)$

When $x = 0$, $\frac{d^2y}{dx^2} = 0$

so $x = 0$ is neither a maximum nor a minimum point

When $x > 0$, $\frac{d^2y}{dx^2} > 0$

When $-2 < x < 0$, $\frac{d^2y}{dx^2} < 0$

$\therefore$ (0, 2) is a stationary point of inflection.

b At points of inflection $\frac{d^2y}{dx^2} = 0$

$e^x x(2 + x) = 0$

$x = 0$ or -2

From part **a** it is known that $x = 0$ is a stationary point of inflection.

When $x < -2$, $\frac{d^2y}{dx^2} > 0$

When $-2 < x < 0$, $\frac{d^2y}{dx^2} < 0$

so $x = -2$ is a point of inflection

$x = -2 \Rightarrow y = 10e^{-2}$

$\therefore$ $(-2, 10e^{-2})$ is a non-stationary point of inflection.

6 a $y = xe^x$

$\frac{dy}{dx} = xe^x + e^x = e^x(x + 1)$

At stationary points $\frac{dy}{dx} = 0$

$e^x(x + 1) = 0$ when $x = -1$ and $y = -e^{-1}$

$\therefore$ stationary point at $\left(-1, -\frac{1}{e}\right)$

$\frac{d^2y}{dx^2} = e^x + e^x(x + 1) = e^x(x + 2)$

When $x = -1$, $\frac{d^2y}{dx^2} = e^{-1} > 0$

Therefore $\left(-1, -\frac{1}{e}\right)$ is a minimum.

b At points of inflection $\frac{d^2y}{dx^2} = 0$

$e^x(x + 2) = 0$

$\Rightarrow x = -2,\ y = -2e^{-2} = -\frac{2}{e^2}$

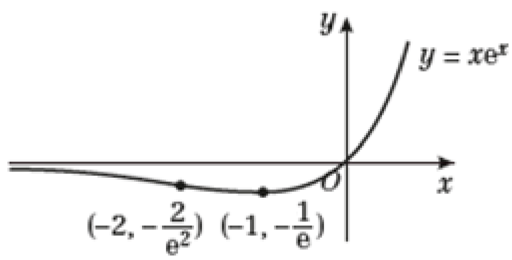
When $x < -2$, $\frac{d^2y}{dx^2} < 0$

When $x > -2$, $\frac{d^2y}{dx^2} > 0$

so $x = -2$ is a point of inflection

$\therefore$ non-stationary point of inflection at $\left(-2, -\frac{2}{e^2}\right)$

c



7 i $f'(x)$ is the gradient, so it is
negative for A
zero for B
positive for C
zero for D

7 ii $f''(x)$ determines whether the curve is convex, is concave or has a point of inflection. Hence $f''(x)$ is
positive for A
positive for B
negative for C
zero for D

8 $f(x) = \tan x$

$f'(x) = \sec^2 x$

$f''(x) = 2\sec^2 x \tan x = 2\frac{\sin x}{\cos^3 x}$

At points of inflection $f''(x) = 0$

$2\frac{\sin x}{\cos^3 x} = 0$ only when $\sin x = 0$, which has only one solution, $x = 0$, in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$

When $x = 0$, $f(x) = 0$

When $x < 0$, $f''(x) < 0$

When $x > 0$, $f''(x) > 0$

$\therefore$ there is one point of inflection at (0, 0).

9 a $y = x(3x-1)^5$

$$\frac{dy}{dx} = 15x(3x-1)^4 + (3x-1)^5$$

$$\frac{d^2y}{dx^2} = 15(3x-1)^4 + 15(3x-1)^4 + 180x(3x-1)^3$$

$$= 30(3x-1)^4 + 180x(3x-1)^3$$

$$= 30(3x-1)^3(9x-1)$$

b At points of inflection $\frac{d^2y}{dx^2} = 0$

$30(3x-1)^3(9x-1) = 0$

$x = \frac{1}{3}$ or $\frac{1}{9}$

$x = \frac{1}{9} \Rightarrow y = \frac{1}{9} \times \left(\frac{1}{3} - 1\right)^5 = -\frac{32}{2187}$

$x = \frac{1}{3} \Rightarrow y = \frac{1}{3} \times (1-1)^5 = 0$

Points of inflection are $\left(\frac{1}{9}, -\frac{3}{2187}\right)$ and $\left(\frac{1}{3}, 0\right)$

10 a $\frac{d^2y}{dx^2} = 12(x-5)^2 \geqslant 0$ for all x, so even though $\frac{d^2y}{dx^2} = 0$ at $x = 5$, the sign of $\frac{d^2y}{dx^2}$ does not change on either side of $x = 5$ and hence it is not a point of inflection.

b $\frac{dy}{dx} = 4(x-5)^3 = 0$ when

$x = 5$ and $y = 0$

Stationary point is at (5, 0).

When $x < 5$, $\frac{dy}{dx} < 0$

When $x > 5$, $\frac{dy}{dx} > 0$

$\therefore$ (5, 0) is a minimum point.

11 $y = \frac{1}{3}x^2 \ln x - 2x + 5$

$$\frac{dy}{dx} = \frac{1}{3}x^2\left(\frac{1}{x}\right) + \frac{2}{3}x\ln x - 2 = \frac{x}{3} + \frac{2}{3}x\ln x - 2$$

$$\frac{d^2y}{dx^2} = \frac{1}{3} + \frac{2}{3}(1 + \ln x) = 1 + \frac{2}{3}\ln x$$

C is convex when $\frac{d^2y}{dx^2} \geqslant 0$

$1 + \frac{2}{3}\ln x \geqslant 0$

$\ln x \geqslant -\frac{3}{2}$

$x \geqslant e^{-\frac{3}{2}}$

Challenge

1 A general cubic can be written as

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f''(x) = 0$ when $x = -\frac{b}{3a}$

Let $\varepsilon > 0$; then

$$f''\left(-\frac{b}{3a} + \varepsilon\right) = 6a\left(-\frac{b}{3a} + \varepsilon\right) + 2b$$
$$= -2b + 6a\varepsilon + 2b = 6a\varepsilon > 0$$
$$f''\left(-\frac{b}{3a} - \varepsilon\right) = 6a\left(-\frac{b}{3a} - \varepsilon\right) + 2b$$
$$= -2b - 6a\varepsilon + 2b = -6a\varepsilon < 0$$

The sign of $f''(x)$ changes either side of $x = -\frac{b}{3a}$, so this is the single point of inflection.

2 a $y = ax^4 + bx^3 + cx^2 + dx + e$

$\frac{dy}{dx} = 4ax^3 + 3bx^2 + 2cx + d$

$\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 2c$

$\frac{d^2y}{dx^2} = 0 \Leftrightarrow 12ax^2 + 6bx + 2c = 0$

As this is a quadratic equation, there are at most two values of x for which $\frac{d^2y}{dx^2} = 0$.

So there are at most two points of inflection.

b If the discriminant of a quadratic is less than zero, there are no real solutions.

$$\text{Discriminant} = (6b)^2 - 4 \times 12a \times 2c$$
$$= 36b^2 - 96ac$$
$$= 12(3b^2 - 8ac)$$

If $3b^2 < 8ac$ then discriminant < 0 and so there are no solutions to $\frac{d^2y}{dx^2} = 0$.

Therefore if $3b^2 < 8ac$, then C has no points of inflection.