Differentiation 9F

1 a $y = \tan 3x$

Using the result

$y = \tan kx \Rightarrow \frac{dy}{dx} = k\sec^2 kx$

$\frac{dy}{dx} = 3\sec^2 3x$

b $y = 4\tan^3 x$

Let $u = \tan x$; then $y = 4u^3$

$\frac{du}{dx} = \sec^2 x$ and $\frac{dy}{du} = 12u^2$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 12u^2 \sec^2 x$$
$$= 12\tan^2 x \sec^2 x$$

c $y = \tan(x-1)$

$\frac{dy}{dx} = \sec^2(x-1)$

d $y = x^2 \tan\frac{1}{2}x + \tan\left(x - \frac{1}{2}\right)$

The first term is a product with

$u = x^2$ and $v = \tan\frac{1}{2}x$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$

Using the product rule for the first term:

$$\frac{dy}{dx} = x^2\left(\frac{1}{2}\sec^2\frac{1}{2}x\right) + \tan\frac{1}{2}x \times 2x + \sec^2\left(x - \frac{1}{2}\right)$$
$$= \frac{1}{2}x^2\sec^2\frac{1}{2}x + 2x\tan\frac{1}{2}x + \sec^2\left(x - \frac{1}{2}\right)$$

2 a $y = \cot 4x$

Let $u = 4x$; then $y = \cot u$

$\frac{du}{dx} = 4$ and $\frac{dy}{du} = -\text{cosec}^2 u$

Using the chain rule,

$\frac{dy}{dx} = -\text{cosec}^2 u \times 4 = -4\,\text{cosec}^2 4x$

b $y = \sec 5x$

Let $u = 5x$; then $y = \sec u$

$\frac{du}{dx} = 5$ and $\frac{dy}{du} = \sec u \tan u$

Using the chain rule,

$\frac{dy}{dx} = 5\sec u\tan u = 5\sec 5x\tan 5x$

c $y = \operatorname{cosec} 4x$

Let $u = 4x$; then $y = \operatorname{cosec} u$

$\frac{du}{dx} = 4$ and $\frac{dy}{du} = -\operatorname{cosec} u \cot u$

Using the chain rule,

$$\frac{dy}{dx} = -4\operatorname{cosec} u \cot u$$
$$= -4\operatorname{cosec} 4x \cot 4x$$

d $y = \sec^2 3x = (\sec 3x)^2$

Let $u = \sec 3x$; then $y = u^2$

$\frac{du}{dx} = 3\sec 3x \tan 3x$ and $\frac{dy}{du} = 2u$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2u \times 3\sec 3x \tan 3x$$
$$= 2\sec 3x \times 3\sec 3x \tan 3x$$
$$= 6\sec^2 3x \tan 3x$$

e $y = x\cot 3x$

This is a product, so let
$u = x$ and $v = \cot 3x$
and use the product rule.

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -3\operatorname{cosec}^2 3x$

$$\frac{dy}{dx} = x(-3\operatorname{cosec}^2 3x) + \cot 3x \times 1$$
$$= \cot 3x - 3x\operatorname{cosec}^2 3x$$

f $y = \frac{\sec^2 x}{x}$

This is a quotient, so let
$u = \sec^2 x$ and $v = x$
and use the quotient rule.

$\frac{du}{dx} = 2\sec x(\sec x \tan x)$ and $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{x(2\sec^2 x \tan x) - \sec^2 x \times 1}{x^2}$$
$$= \frac{\sec^2 x(2x\tan x - 1)}{x^2}$$

g $y = \operatorname{cosec}^3 2x$

Let $u = \operatorname{cosec} 2x$; then $y = u^3$

$\frac{du}{dx} = -2\operatorname{cosec} 2x \cot 2x$ and $\frac{dy}{du} = 3u^2$

Using the chain rule,

$$\frac{dy}{dx} = 3u^2(-2\operatorname{cosec} 2x \cot 2x)$$
$$= -6\operatorname{cosec}^2 2x \operatorname{cosec} 2x \cot 2x$$
$$= -6\operatorname{cosec}^3 2x \cot 2x$$

h $y = \cot^2(2x-1)$

Let $u = \cot(2x-1)$; then $y = u^2$

$\frac{du}{dx} = -2\operatorname{cosec}^2(2x-1)$ and $\frac{dy}{du} = 2u$

Using the chain rule,

$$\frac{dy}{dx} = 2u\left(-2\operatorname{cosec}^2(2x-1)\right)$$
$$= -4\cot(2x-1)\operatorname{cosec}^2(2x-1)$$

3 a $f(x) = (\sec x)^{\frac{1}{2}}$

Using the chain rule,

$$f'(x) = \frac{1}{2}(\sec x)^{-\frac{1}{2}} \times \sec x \tan x$$
$$= \frac{1}{2}(\sec x)^{\frac{1}{2}} \tan x$$

3 b $f(x)=\sqrt{\cot x}=(\cot x)^{\frac{1}{2}}$

Using the chain rule,

$$f'(x)=\frac{1}{2}(\cot x)^{-\frac{1}{2}}\times(-\operatorname{cosec}^2 x)$$
$$=-\frac{1}{2}(\cot x)^{-\frac{1}{2}}\operatorname{cosec}^2 x$$

c $f(x)=\operatorname{cosec}^2 x=(\operatorname{cosec} x)^2$

Using the chain rule,

$$f'(x)=2(\operatorname{cosec} x)^1(-\operatorname{cosec} x\cot x)$$
$$=-2\operatorname{cosec}^2 x\cot x$$

d $f(x)=\tan^2 x=(\tan x)^2$

Using the chain rule,

$$f'(x)=2\tan x\times\sec^2 x=2\tan x\sec^2 x$$

e $f(x)=\sec^3 x=(\sec x)^3$

Using the chain rule,

$$f'(x)=3(\sec x)^2\sec x\tan x=3\sec^3 x\tan x$$

f $f(x)=\cot^3 x=(\cot x)^3$

Using the chain rule,

$$f'(x)=3(\cot x)^2(-\operatorname{cosec}^2 x)$$
$$=-3\cot^2 x\operatorname{cosec}^2 x$$

4 a $f(x)=x^2\sec 3x$

Let $u=x^2$ and $v=\sec 3x$

$\frac{du}{dx}=2x$ and $\frac{dv}{dx}=3\sec 3x\tan 3x$

Using the product rule,

$$f'(x)=3x^2\sec 3x\tan 3x+2x\sec 3x$$

b $f(x)=\frac{\tan 2x}{x}$

Let $u=\tan 2x$ and $v=x$

$\frac{du}{dx}=2\sec^2 2x$ and $\frac{dv}{dx}=1$

Using the quotient rule,

$$f'(x)=\frac{2x\sec^2 2x-\tan 2x}{x^2}$$

c $f(x)=\frac{x^2}{\tan x}$

Let $u=x^2$ and $v=\tan x$

$\frac{du}{dx}=2x$ and $\frac{dv}{dx}=\sec^2 x$

Using the quotient rule,

$$f'(x)=\frac{2x\tan x-x^2\sec^2 x}{\tan^2 x}$$

d $f(x)=e^x\sec 3x$

Let $u=e^x$ and $v=\sec 3x$

$\frac{du}{dx}=e^x$ and $\frac{dv}{dx}=3\sec 3x\tan 3x$

Using the product rule,

$$f'(x)=3e^x\sec 3x\tan 3x+e^x\sec 3x$$
$$=e^x\sec 3x(3\tan 3x+1)$$

e $f(x)=\frac{\ln x}{\tan x}$

Let $u=\ln x$ and $v=\tan x$

$\frac{du}{dx}=\frac{1}{x}$ and $\frac{dv}{dx}=\sec^2 x$

Using the quotient rule,

$$f'(x)=\frac{\left(\frac{1}{x}\right)\tan x-\ln x\sec^2 x}{\tan^2 x}$$
$$=\frac{\tan x-x\ln x\sec^2 x}{x\tan^2 x}$$

f $f(x)=\frac{e^{\tan x}}{\cos x}$

Let $u=e^{\tan x}$ and $v=\cos x$

$\frac{du}{dx}=e^{\tan x}\sec^2 x$ and $\frac{dv}{dx}=-\sin x$

Using the quotient rule,

$$f'(x)=\frac{e^{\tan x}\sec^2 x\cos x-e^{\tan x}(-\sin x)}{\cos^2 x}$$
$$=\frac{e^{\tan x}\sec x+e^{\tan x}\sin x}{\cos^2 x}$$
$$=\frac{e^{\tan x}(\sec x+\sin x)}{\cos^2 x}$$
$$=e^{\tan x}(\sec^3 x+\sec x\tan x)$$
$$=e^{\tan x}\sec x(\sec^2 x+\tan x)$$

5 a $y = \dfrac{1}{\cos x \sin x} = \sec x \operatorname{cosec} x$

Let $u = \sec x$ and $v = \operatorname{cosec} x$

$\dfrac{du}{dx} = \sec x \tan x$ and $\dfrac{dv}{dx} = -\operatorname{cosec} x \cot x$

Using the product rule,

$$\frac{dy}{dx} = \sec x(-\operatorname{cosec} x \cot x) + \operatorname{cosec} x(\sec x \tan x)$$
$$= -\frac{\cos x}{\cos x \sin x \sin x} + \frac{\sin x}{\sin x \cos x \cos x}$$
$$= -\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}$$

Alternative solution:

$y = \dfrac{1}{\cos x \sin x} = \dfrac{2}{\sin 2x} = 2\operatorname{cosec} 2x$

(because $\sin 2x = 2\sin x \cos x$)

$\dfrac{dy}{dx} = -4\operatorname{cosec} 2x \cot 2x$

b At stationary points $\dfrac{dy}{dx} = 0$

$$\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x} = 0$$
$$\frac{1}{\cos^2 x} = \frac{1}{\sin^2 x}$$
$$\tan^2 x = 1$$
$$\tan x = \pm 1$$

In the interval $0 < x \leqslant \pi$

there are two solutions, $\dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$.

So the number of stationary points is 2.

Alternative solution:

$-4\operatorname{cosec} 2x \cot 2x = 0$

$\operatorname{cosec} 2x \neq 0$

but $\cot 2x = 0$ has two solutions

in the interval $0 < x \leqslant \pi$.

So there are 2 stationary points.

c When $x = \dfrac{\pi}{3}$,

$$y = \frac{1}{\cos\frac{\pi}{3}\sin\frac{\pi}{3}} = \frac{1}{\frac{1}{2}\times\frac{\sqrt{3}}{2}} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

$$\frac{dy}{dx} = -\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\left(\frac{1}{2}\right)^2} = -\frac{4}{3} + 4 = \frac{8}{3}$$

or, using the alternative expression,

$$\frac{dy}{dx} = -4\times\frac{2}{\sqrt{3}}\times\left(-\frac{1}{\sqrt{3}}\right) = \frac{8}{3}$$

Equation of tangent is

$$y - \frac{4\sqrt{3}}{3} = \frac{8}{3}\left(x - \frac{\pi}{3}\right)$$
$$3y - 4\sqrt{3} = 8x - \frac{8\pi}{3}$$
$$24x - 9y + 12\sqrt{3} - 8\pi = 0$$

This is in the required form $ax + by + c = 0$

With $a = 24$, $b = -9$ and $c = 12\sqrt{3} - 8\pi$.

6 $y = \sec x = \dfrac{1}{\cos x}$

Let $u = 1$ and $v = \cos x$

$\dfrac{du}{dx} = 0$ and $\dfrac{dv}{dx} = -\sin x$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{\cos x \times 0 - 1\times(-\sin x)}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

7 $y = \cot x = \dfrac{1}{\tan x}$

Let $u = 1$ and $v = \tan x$

$\dfrac{du}{dx} = 0$ and $\dfrac{dv}{dx} = \sec^2 x$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{\tan x \times 0 - 1\times \sec^2 x}{\tan^2 x} = -\frac{\sec^2 x}{\tan^2 x}$$
$$= -\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

8 $x = \cos 2y$

$$\frac{dx}{dy} = -2\sin 2y$$

$$\frac{dy}{dx} = \frac{-1}{2\sin 2y}$$

$$\sin^2 2y + \cos^2 2y = 1$$

$$\sin 2y = \sqrt{1-\cos^2 2y}$$

$$= \sqrt{1-x^2}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}$$

9 a $x = \text{cosec } 5y$

$$\frac{dx}{dy} = \frac{d}{dy}(5y)\bullet(-\cot 5y \text{ cosec} 5y)$$

$$\frac{dx}{dy} = -5\cot 5y \text{ cosec} 5y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{-1}{5\cot 5y \text{ cosec} 5y}$$

b $\frac{dy}{dx} = \frac{-1}{5\cot 5y \text{ cosec} 5y}$

$x = \text{cosec } 5y$

$$\cot^2 5y + 1 = \text{cosec}^2 5y$$

$$\Rightarrow \cot 5y = \sqrt{\text{cosec}^2 5y - 1} = \sqrt{x^2 - 1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{5x\sqrt{x^2-1}}$$

Challenge

a Let $y = \arccos x$

So $x = \cos y$

$$\frac{dx}{dy} = -\sin y$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

b Let $y = \arctan x$

So $x = \tan y$

$$\frac{dx}{dy} = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$