Differentiation 9A

1 a $f(x)=\cos x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$
$$=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$
$$=\lim_{h\to 0}\left(\left(\frac{\cos h-1}{h}\right)\cos x-\frac{\sin h}{h}\sin x\right)$$

b Since $\frac{\cos h-1}{h}\to 0$ and $\frac{\sin h}{h}\to 1$

the expression inside the limit in part **a** tends to $0\times\cos x-1\times\sin x=-\sin x$

So $f'(x)=-\sin x$

2 a $y=2\cos x$

$$\frac{dy}{dx}=2\times(-\sin x)=-2\sin x$$

b $y=2\sin\frac{1}{2}x$

$$\frac{dy}{dx}=2\times\frac{1}{2}\cos\frac{1}{2}x=\cos\frac{1}{2}x$$

c $y=\sin 8x$

$$\frac{dy}{dx}=8\cos 8x$$

d $y=6\sin\frac{2}{3}x$

$$\frac{dy}{dx}=6\times\frac{2}{3}\cos\frac{2}{3}x=4\cos\frac{2}{3}x$$

3 a $f(x)=2\cos x$

$$f'(x)=2\times(-\sin x)=-2\sin x$$

b $f(x)=6\cos\frac{5}{6}x$

$$f'(x)=6\times\left(-\frac{5}{6}\sin\frac{5}{6}x\right)=-5\sin\frac{5}{6}x$$

c $f(x)=4\cos\frac{1}{2}x$

$$f'(x)=4\times\left(-\frac{1}{2}\sin\frac{1}{2}x\right)=-2\sin\frac{1}{2}x$$

d $f(x)=3\cos 2x$

$$f'(x)=3(-2\sin 2x)=-6\sin 2x$$

4 a $y=\sin 2x+\cos 3x$

$$\frac{dy}{dx}=2\cos 2x+(-3\sin 3x)$$
$$=2\cos 2x-3\sin 3x$$

b $y=2\cos 4x-4\cos x+2\cos 7x$

$$\frac{dy}{dx}=2\times(-4\sin 4x)-4\times(-\sin x)$$
$$+2\times(-7\sin 7x)$$
$$=-8\sin 4x+4\sin x-14\sin 7x$$

c $y=x^2+4\cos 3x$

$$\frac{dy}{dx}=2x+4(-3\sin 3x)=2x-12\sin 3x$$

d $y=\frac{1+2x\sin 5x}{x}=\frac{1}{x}+2\sin 5x$

$$\frac{dy}{dx}=-\frac{1}{x^2}+2\times(5\cos 5x)$$
$$=-\frac{1}{x^2}+10\cos 5x$$

5 $y = x - \sin 3x$

$\frac{dy}{dx} = 1 - 3\cos 3x$

At stationary points $\frac{dy}{dx} = 0$

$1 - 3\cos 3x = 0$

$\cos 3x = \frac{1}{3}$

$3x = 1.23..., 5.05...$ or $7.51...$

$x = 0.410, 1.68$ or 2.50 (3 s.f.)

$x = 0.410 \Rightarrow y = 0.41 - \sin 1.23 = -0.532$
$x = 1.68 \Rightarrow y = 1.68 - \sin 5.04 = 2.63$
$x = 2.50 \Rightarrow y = 2.50 - \sin 7.50 = 1.56$

Stationary points in the interval $0 \le x \le \pi$ are (0.410, −0.532), (1.68, 2.63) and (2.50, 1.56).

6 $y = 2\sin 4x - 4\cos 2x$

$\frac{dy}{dx} = 2 \times 4\cos 4x - 4 \times (-2\sin 2x)$

$= 8\cos 4x + 8\sin 2x$

When $x = \frac{\pi}{2}$:

$\frac{dy}{dx} = 8\cos 2\pi + 8\sin \pi$

$= 8 \times 1 + 8 \times 0 = 8$

So the gradient of the curve at the point where $x = \frac{\pi}{2}$ is 8.

7 $y = 2\sin 2x + \cos 2x$

$\frac{dy}{dx} = 2 \times 2\cos 2x + (-2\sin 2x)$

$= 4\cos 2x - 2\sin 2x$

At stationary points $\frac{dy}{dx} = 0$

$4\cos 2x - 2\sin 2x = 0$
$4 - 2\tan 2x = 0$
$\tan 2x = 2$
$2x = 1.107...$ or $4.248...$
$x = 0.554$ or 2.12 (3 s.f.)

When $x = 0.554$:
$y = 2\sin(2 \times 0.554) + \cos(2 \times 0.554) = 2.24$
When $x = 2.12$:
$y = 2\sin(2 \times 2.12) + \cos(2 \times 2.12) = -2.24$

Stationary points in the interval $0 \leqslant x \leqslant \pi$ are (0.554, 2.24) and (2.12, −2.24).

8 $y = \sin 5x + \cos 3x$

$\frac{dy}{dx} = 5\cos 5x - 3\sin 3x$

At $(\pi, -1)$, $\frac{dy}{dx} = 5\cos 5\pi - 3\sin 3\pi$

$= 5 \times (-1) - 3 \times 0 = -5$

Equation of tangent is $y - (-1) = -5(x - \pi)$
or $y = -5x + 5\pi - 1$

9 $y = 2x^2 - \sin x$

$\frac{dy}{dx} = 4x - \cos x$

When $x = \pi$, $y = 2\pi^2$ and

$\frac{dy}{dx} = 4\pi - \cos \pi = 4\pi + 1$

Gradient of normal is $-\frac{1}{4\pi + 1}$

Equation of normal is

$y - 2\pi^2 = -\frac{1}{4\pi + 1}(x - \pi)$

Multiplying through by $(4\pi + 1)$ and rearranging gives
$x + (4\pi + 1)y - \pi(8\pi^2 + 2\pi + 1) = 0$

10 Let $f(x) = \sin x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \right)$$

$$= \lim_{h \to 0} \left(\left(\frac{\cos h - 1}{h} \right) \sin x + \left(\frac{\sin h}{h} \right) \cos x \right)$$

Since $\frac{\cos h - 1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$,

the expression inside the limit tends to

$(0 \times \sin x + 1 \times \cos x)$

So $\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$

Hence the derivative of $\sin x$ is $\cos x$.

Challenge

Let $f(x) = \sin(kx)$

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\sin(kx + kh) - \sin(kx)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\sin kx \cos kh + \cos kx \sin kh - \sin kx}{h} \right)$$

$$= \lim_{h \to 0} \left(\left(\frac{\cos kh - 1}{h} \right) \sin kx + \left(\frac{\sin kh}{h} \right) \cos kx \right)$$

As $h \to 0$, $\left(\frac{\sin kh}{h} \right) \to k$ and $\left(\frac{\cos kh - 1}{h} \right) \to 0$,

so the expression inside the limit tends to

$0 \times \sin kx + k \times \cos kx = k \cos kx$

Hence the derivative of $\sin(kx)$ is $k \cos(kx)$.