At $A$, $y = 0 \Rightarrow 3\sin t = 0 \Rightarrow \sin t = 0$

So $t = 0$ or $t = \pi$

Substitute $t = 0$ and $t = \pi$ into

$x = 4 \cos t$

$t = 0 \Rightarrow x = 4 \cos 0 = 4 \times 1 = 4$
$t = \pi \Rightarrow x = 4 \cos \pi = 4 \times (-1) = -4$

The coordinates of $A$ are $(4, 0)$.

At $B$, $x = 0 \Rightarrow 4 \cos t = 0 \Rightarrow \cos t = 0$

So $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$

Substitute $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ into

$y = 3 \sin t$

$t = \frac{\pi}{2} \Rightarrow y = 3 \sin \frac{\pi}{2} = 3 \times 1 = 3$
$t = \frac{3\pi}{2} \Rightarrow y = 3 \sin \frac{3\pi}{2} = 3 \times (-1) = -3$

The coordinates of $B$ are $(0, 3)$.

At $C$, $t = \frac{\pi}{6}$

$x = 4 \cos \frac{\pi}{6} = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$y = 3 \sin \frac{\pi}{6} = 3 \times \frac{1}{2} = \frac{3}{2}$

The coordinates of $C$ are $\left(2\sqrt{3}, \frac{3}{2}\right)$.

c \quad $x = 4 \cos t \Rightarrow \frac{x}{4} = \cos t \quad (1)$

$y = 3 \sin t \Rightarrow \frac{y}{3} = \sin t \quad (2)$

Substitute (1) and (2) into

$\cos^2 t + \sin^2 t = 1$

\[
\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1
\]

\[
\frac{x^2}{16} + \frac{y^2}{9} = 1
\]

$9x^2 + 16y^2 = 144$

2 Substitute $t = 0$ into

$x = \cos t$ and $y = \frac{1}{2} \sin 2t$

$x = \cos 0 = 1$

$y = \frac{1}{2} \sin (2 \times 0) = \frac{1}{2} \sin 0 = \frac{1}{2} \times 0 = 0$

So when $t = 0$, $(x, y) = (1, 0)$.

Substitute $t = \frac{\pi}{2}$ into

$x = \cos t$ and $y = \frac{1}{2} \sin 2t$

$x = \cos \frac{\pi}{2} = 0$

$y = \frac{1}{2} \sin \left(2 \times \frac{\pi}{2}\right) = \frac{1}{2} \sin \pi = \frac{1}{2} \times 0 = 0$

So when $t = \frac{\pi}{2}$, $(x, y) = (0, 0)$.

Substitute $t = \pi$ into

$x = \cos t$ and $y = \frac{1}{2} \sin 2t$

$x = \cos \pi = -1$

$y = \frac{1}{2} \sin 2\pi = \frac{1}{2} \times 0 = 0$

So when $t = \pi$, $(x, y) = (-1, 0)$.

Substitute $t = 3\pi$ into

$x = \cos t$ and $y = \frac{1}{2} \sin 2t$

$x = \cos \frac{3\pi}{2} = 0$

$y = \frac{1}{2} \sin \left(2 \times \frac{3\pi}{2}\right) = \frac{1}{2} \sin 3\pi = \frac{1}{2} \times 0 = 0$

So when $t = \frac{3\pi}{2}$, $(x, y) = (0, 0)$. 

![Graph](Image)
3 a \[ x = e^{2t} + 1 \] \[ x - 1 = e^{2t} \] \[ \ln(x - 1) = 2t + 1 \] \[ \ln(x - 1) - 1 = 2t \] \[ \frac{1}{2} \ln(x - 1) - \frac{1}{2} = t \] \[ y = t + \ln 2 \] \[ \text{(2)} \]

Substitute (2) into (3).
\[ y = \frac{1}{2} \ln(x - 1) - \frac{1}{2} + \ln 2 \]
\[ = \ln\left(\frac{x - 1}{2}\right) + \ln 2 - \frac{1}{2} \]
\[ = \ln(2\sqrt{x - 1}) - \frac{1}{2} \]

The Cartesian equation of the curve is
\[ y = \ln(2\sqrt{x - 1}) - \frac{1}{2} \]

Substitute \( t = 1 \) into (1).
\[ x = e^3 + 1 \]
Since \( x \) is an increasing function and \( t > 1 \),
\[ x > e^3 + 1 \]
So \( k = e^3 + 1 \)

b The range of \( f(x) \) is the range of \( y = q(t) \) so substitute \( t = 1 \) into (3).
\[ y = 1 + \ln 2 \]
Since \( y \) is an increasing function and \( t > 1 \),
\[ y > 1 + \ln 2 \]
The range of \( f(x) \) is \( y > 1 + \ln 2 \).

4 \[ x = \frac{1}{2t + 1} \]
\[ 2t + 1 = \frac{1}{x} \]
\[ 2t = \frac{1}{x} - 1 \]
\[ t = \frac{1}{2x} - \frac{1}{2} \] \[ \text{(2)} \]

\[ y = 2 \ln\left(t + \frac{1}{2}\right) \] \[ \text{(3)} \]

Substitute (2) into (3).
\[ y = 2 \ln\left(\frac{1}{2x} - \frac{1}{2} + \frac{1}{2}\right) \]
\[ = 2 \ln\left(\frac{1}{2x}\right) \]
\[ = -2 \ln(2x) \]
\[ = -2(\ln 2 + \ln x) \]
\[ = -\ln 4 - 2 \ln x \]

The Cartesian equation of the curve is
\[ y = -\ln 4 - 2 \ln x \]

The domain of \( f(x) \) is the range of \( x = p(t) \) so substitute \( t = \frac{1}{2} \) into (1).
\[ x = \frac{1}{2} \left(\frac{1}{2}\right) + 1 = \frac{1}{2} \]
As \( t \to \infty \), \( x \to 0 \).
So the domain is \( 0 < x < \frac{1}{2} \).

The range of \( f(x) \) is the range of \( y = q(t) \) so substitute \( t = \frac{1}{2} \) into (3).
\[ y = 2 \ln\left(\frac{1}{2} + \frac{1}{2}\right) = 0 \]
As \( t \to \infty \), \( y \to \infty \)
So the range is \( y > 0 \).
5 a \[ x = \sin t \quad \frac{1}{1 + t} \]  \[ y = \cos^2 t \]  \[ x(1 + t) = 1 \]

Substitute (1) and (2) into
\[ \cos^2 t = 1 - 2 \sin^2 t \]
\[ y = 1 - 2x^2 \]

A Cartesian equation of the curve is \[ y = 1 - 2x^2 \]

b Substitute \( y = 0 \) into
\[ y = 1 - 2x^2 \]
\[ 0 = 1 - 2x^2 \]
\[ 2x^2 = 1 \]
\[ x^2 = \frac{1}{2} \]
\[ x = \pm \sqrt{\frac{1}{2}} = \pm \frac{\sqrt{2}}{2} \]
\[ x = \pm \frac{\sqrt{2}}{2} = \pm \frac{\sqrt{2}}{2} \]

So the curve meets the \( x \)-axis at \( \left( \frac{\sqrt{2}}{2}, 0 \right) \) and \( \left( -\frac{\sqrt{2}}{2}, 0 \right) \).

So \( a \) and \( b \) are \( \frac{\sqrt{2}}{2} \) and \( -\frac{\sqrt{2}}{2} \).

6 \[ x = \frac{1}{1 + t} \]

Substitute (1) into
\[ x(1 + t) = 1 \]
\[ 1 + t = \frac{1}{x} \]
So \( t = \frac{1}{x} - 1 \)

Substitute (2) into
\[ y = \frac{1}{(1 + t)(1 - t)} \]
\[ y = \frac{1}{(1 + \frac{1}{x} - 1)(1 - (\frac{1}{x} - 1))} \]
\[ y = \frac{1}{\left( \frac{1}{x} \right) (2 - \frac{1}{x})} \]
\[ y = \frac{x^2}{2x - 1} \]

So the Cartesian equation of the curve is \[ y = \frac{x^2}{2x - 1} \]

7 a \[ x = 4 \sin t - 3 \Rightarrow \sin t = \frac{x + 3}{4} \quad \frac{1}{4} \]
\[ y = 4 \cos t + 5 \Rightarrow \cos t = \frac{y - 5}{4} \]

Substitute (1) and (2) into
\[ \sin^2 t + \cos^2 t = 1 \]
\[ \left( \frac{x + 3}{4} \right)^2 + \left( \frac{y - 5}{4} \right)^2 = 1 \]
\[ \frac{(x + 3)^2}{16} + \frac{(y - 5)^2}{16} = 1 \]
\[ (x + 3)^2 + (y - 5)^2 = 16 \]
7 b The circle \((x + 3)^2 + (y - 5)^2 = 4^2\) has centre \((-3, 5)\) and radius 4.

The points of intersection of the circle and the y-axis are at \((0, 5 + \sqrt{7})\) and \((0, 5 - \sqrt{7})\).

c Substitute \(x = 0\) into \((x + 3)^2 + (y - 5)^2 = 16\)

\[
(x + 3)^2 + (0 - 5)^2 = 16
\]

\[
3^2 + (y - 5)^2 = 16
\]

\[
9 + (y - 5)^2 = 16
\]

\[
(y - 5)^2 = 7
\]

\[
y - 5 = \pm \sqrt{7}
\]

\[
y = 5 \pm \sqrt{7}
\]

8 a \[
x = \frac{2 - 3t}{1 + t}
\]

\[
x + xt = 2 - 3t
\]

\[
xt + 3t = 2 - x
\]

\[
t(x + 3) = 2 - x
\]

\[
t = \frac{2 - x}{x + 3}
\]

\[
y = \frac{3 + 2t}{1 + t}
\]

Substitute (2) into (3).

\[
y = \frac{3 + 2\left(\frac{2 - x}{x + 3}\right)}{1 + \left(\frac{2 - x}{x + 3}\right)}
\]

\[
y = \frac{3(x + 3) + 2(2 - x)}{x + 3}
\]

\[
y = \frac{3x + 9 + 4 - 2x}{x + 3}
\]

\[
y = \frac{x + 13}{5}
\]

This is in the form \(y = mx + c\), therefore the curve \(C\) is a straight line.
8 b Substitute $t = 0$ into (1) and (2).

$$x = \frac{2 - 3(0)}{1 + 0} = 2$$
$$y = \frac{3 + 2(0)}{1 + 0} = 3$$

Coordinates are $(2, 3)$.

Substitute $t = 4$ into (1) and (2).

$$x = \frac{2 - 3(4)}{1 + 4} = -2$$
$$y = \frac{3 + 2(4)}{1 + 4} = \frac{11}{5}$$

Coordinates are $\left(-2, \frac{11}{5}\right)$.

Length $= \sqrt{(2 - (-2))^2 + \left(3 - \frac{11}{5}\right)^2}$

$= \sqrt{4^2 + \left(\frac{4}{5}\right)^2}$

$= \sqrt{\frac{416}{25}}$

$= \frac{4\sqrt{26}}{5}$

9 a $x = t^2 - 2$

$x + 2 = t^2$

$\pm \sqrt{x + 2} = t$

But $0 \leq t \leq 2$ so choose the positive value.

$$t = \sqrt{x + 2} \hspace{1cm} (1)$$

Substitute (1) into

$$y = 2t$$

$$y = 2\sqrt{x + 2}$$

b Domain of $f(x)$ is $-2 \leq x \leq 2$.

Range of $f(x)$ is $0 \leq y \leq 4$.

c Substituting (1) and (2) into

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y + 5}{2}\right)^2 = 1$$

$$x^2 + (y + 5)^2 = 4$$

So the curve $C$ forms part of a circle of radius 2 and centre $(0, -5)$.

10 a $x = 2\cos t \Rightarrow \frac{x}{2} = \cos t \hspace{1cm} (1)$

$$y = 2\sin t - 5 \Rightarrow \frac{y + 5}{2} = \sin t \hspace{1cm} (2)$$

Substitute (1) and (2) into

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y + 5}{2}\right)^2 = 1$$

$$x^2 + (y + 5)^2 = 4$$

So the curve $C$ forms part of a circle.

b Since $0 \leq t \leq \pi$, the curve $C$ forms half of the circle.

Arc length $= r\theta = 2\pi$
11 a $x = t - 2 \Rightarrow x + 2 = t$ \hfill (1)

Substitute (1) into

$y = t^3 - 2t^2$

$y = (x + 2)^3 - 2(x + 2)^2$

$= (x + 2)^2(x + 2 - 2)$

$= x(x^2 + 4x + 4)$

$= x^3 + 4x^2 + 4x$

The Cartesian equation of $C$ is $y = x^3 + 4x^2 + 4x$

b

12 $x = t - 3$

$y = 4 - t^2$ \hfill (2)

Substitute (1) and (2) into

$y = 4x + 20$

$4 - t^2 = 4(t - 3) + 20$

$0 = t^2 + 4t + 4$

Use the discriminant:

$b^2 - 4ac = 4^2 - 4(1)(4) = 0$

Therefore, the line and the curve have only one point of intersection, that is, they touch. So the line is a tangent to the curve.

13 a $x = 2 \ln t$ \hfill (1)

$x = 5$ \hfill (2)

Substitute (2) into (1).

$5 = 2 \ln t$

$\frac{5}{2} = \ln t$

$e^{\frac{5}{2}} = t$ \hfill (3)

Substitute (3) into

$y = t^2 - 1$

$y = \left(e^{\frac{5}{2}}\right)^2 - 1$

$= e^5 - 1$

The coordinates of the point of intersection of the line $x = 5$ and the curve are $\left(5, e^5 - 1\right)$.

b $y = t^2 - 1, \; t > 0$

(In this domain) the function is increasing. So the range is $y > -1$.

Therefore, $k > -1$. 
At $A$, $x = 0$

\[ x = 1 + 2t \]
\[ 0 = 1 + 2t \]
\[ -\frac{1}{2} = t \]

So substitute $t = -\frac{1}{2}$ into

\[ y = 4^t - 1 \]
\[ y = 4^{-\frac{1}{2}} - 1 = -\frac{1}{2} \]

The coordinates of $A$ are $\left(0, -\frac{1}{2}\right)$.

At $B$, $y = 0$

\[ y = 4^t - 1 \]
\[ 0 = 4^t - 1 \]
\[ 1 = 4^t \]
\[ t = 0 \]

So substitute $t = 0$ into (1).

\[ x = 1 + 2t \]
\[ x = 1 + 2(0) = 1 \]

The coordinates of $B$ are $\left(1, 0\right)$.

\[ b \]

\[ m_1 = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ = \frac{0 - \left(-\frac{1}{2}\right)}{1 - 0} = \frac{1}{2} \]

Substitute $\left(1, 0\right)$ and $m = \frac{1}{2}$ into

\[ y - y_1 = m(x - x_1) \]
\[ y - 0 = \frac{1}{2}(x - 1) \]
\[ 2y = x - 1 \]
\[ x - 2y - 1 = 0 \]

At $A$, $x = 0$

\[ x = \ln t - \ln \left(\frac{\pi}{2}\right) \quad (1) \]
\[ 0 = \ln t - \ln \left(\frac{\pi}{2}\right) \]

\[ \ln \left(\frac{\pi}{2}\right) = \ln t \]
\[ \frac{\pi}{2} = t \]
\[ y = \sin t \]

(2)

Substitute $t = \frac{\pi}{2}$ into (2).

\[ y = \sin \frac{\pi}{2} = 1 \]

The coordinates of $A$ are $\left(0, 1\right)$.

At $B$, $y = 0$

\[ y = \sin t \]
\[ 0 = \sin t \], $0 < t < 2\pi$
\[ \pi = t \]

Substitute $t = \pi$ into (1).

\[ x = \ln \pi - \ln \left(\frac{\pi}{2}\right) \]
\[ = \ln \pi - \left(\ln \pi - \ln 2\right) = \ln 2 \]

The coordinates of $B$ are $\left(\ln 2, 0\right)$.

\[ m_1 = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ = \frac{0 - 1}{\ln 2 - 0} = -\frac{1}{\ln 2} \]

Substitute $\left(0, 1\right)$ and $m = -\frac{1}{\ln 2}$ into

\[ y - y_1 = m(x - x_1) \]
\[ y - 1 = -\frac{1}{\ln 2}(x - 0) \]
\[ y \ln 2 - x = -x \]
\[ x + y \ln 2 - \ln 2 = 0 \]
16 a  \[ x = 80t \]  \[ \frac{x}{80} = t \]  \[ y = 3000 - 30t \]  \[ \text{Substitute (2) into (3).} \]
\[ y = 3000 - 30\left(\frac{x}{80}\right) \]
\[ = 3000 - \frac{3}{8}x \]
This is in the form \( y = mx + c \), therefore the plane’s descent is a straight line.

b Substitute \( y = 30 \) into (3).
\[ 30 = 3000 - 30t \]
\[ 1 = 100 - t \]
\[ t = 99 \]
So \( k = 99 \)

c Substitute \( t = 99 \) into (1).
\[ x = 80(99) = 7920 \]
During this portion of its descent the plane travels 7920 m horizontally and descends vertically by 3000 – 30 = 2970 m.

Distance travelled \[ = \sqrt{7920^2 + 2970^2} \]
\[ = 8458.56 \text{ m (2 d.p.)} \]

17 a Substitute \( y = 0 \) into
\[ y = 1.5 - 4.9t^2 + 50\sqrt{2}t \]
\[ 0 = 1.5 - 4.9t^2 + 50\sqrt{2}t \]
\[ 4.9t^2 - 50\sqrt{2}t - 1.5 = 0 \]
\[ t = \frac{50\sqrt{2} \pm \sqrt{2(50)^2 - 4(4.9)(-1.5)}}{2(4.9)} \]
\[ = 14.45 \ldots \text{ or } t = -0.021 \ldots \]
\[ t = 14.45 \text{ (since } t > 0) \]
\[ x = 50\sqrt{2}t \]
Substitute \( t = 14.45 \) into (2).
\[ x = 50\sqrt{2}(14.45\ldots) \]
\[ = 1022 \ldots \]
The furthest horizontal distance is 1022 m (4 s.f.).

b Substitute \( x = 1000 \) into (2).
\[ 1000 = 50\sqrt{2}t \]
\[ t = \frac{1000}{50\sqrt{2}} \]
\[ = \frac{20}{\sqrt{2}} = 10\sqrt{2} \]

Substitute \( t = 10\sqrt{2} \) into (1).
\[ y = 1.5 - 4.9\left(10\sqrt{2}\right)^2 + 50\sqrt{2}\left(10\sqrt{2}\right) \]
\[ = 1.5 - 490 \times 2 + 500 \times 2 \]
\[ = 1.5 + 20 \]
\[ = 21.5 \]
21.5 > 10, so the arrow is too high to hit the castle wall.

c Substitute \( y = 10 \) into (1).
\[ 10 = 1.5 - 4.9t^2 + 50\sqrt{2}t \]
\[ 4.9t^2 - 50\sqrt{2}t + 8.5 = 0 \]
\[ t = \frac{50\sqrt{2} \pm \sqrt{2(50)^2 - 4(4.9)(8.5)}}{2(4.9)} \]
\[ = 14.309 \ldots \text{ (or } t = 0.1212\ldots) \]
\[ x = 50\sqrt{2}(14.309\ldots) = 1011.799\ldots \]
The archer needs to move back by 11.8 m (3 s.f.).

18 a \[ y = 244(t(4-t)) \text{, } 0 < t < 4 \] \[ \text{ is a parabola with midpoint (which corresponds to the maximum height) at } t = 2 \text{ hours.} \]
Substitute \( t = 2 \) into (1).
\[ y = 244(2)(4 - 2) = 976 \text{ m} \]

b The mountaineer completes her walk at sea level, when \( y = 0 \).
\[ y = 0 \text{ when } t = 4 \text{ (and when } t = 0) \]
Substitute \( t = 4 \) into
\[ x = 300\sqrt{t} \]
\[ x = 300\sqrt{4} = 600 \text{ m} \]
The horizontal distance is 600 m.
19 a The curve is symmetrical in the x-axis. Its highest point is at the midpoint, at which \( t = \pi \).

At \( t = \pi \):
\[
y = -\cos t
\]
\[
= -\cos \pi
\]
\[
= 1
\]

The height of the bridge is 10 m.

b At \( t = \frac{\pi}{2} \):
\[
x = \frac{4t}{\pi} - 2\sin t
\]
\[
= \frac{4}{\pi} \times \frac{\pi}{2} - 2\sin \frac{\pi}{2}
\]
\[
= 0
\]

At \( t = \frac{3\pi}{2} \):
\[
x = \frac{4}{\pi} \times \frac{3\pi}{2} - 2\sin \frac{3\pi}{2}
\]
\[
= 8
\]

The maximum width is \( 80 - 0 = 80 \) m.

20 a \[ y = 10(t - 1)^2 \] (1)

Substitute \( t = 0 \) into (1).
\[
y = 10(0 - 1)^2 = 10
\]

The cyclist’s initial height is 10 m.

b Substitute \( y = 0 \) into (1).
\[
0 = 10(t - 1)^2
\]
\[
t = 1
\]

The cyclist is at her lowest height after 1 second.

c Substitute \( t = 1.3 \) into (1).
\[
y = 10(1.3 - 1)^2 = 0.9
\]

The cyclist leaves the ramp at height 0.9 m.
**Challenge**

**a** If the particles collide at time $t$ seconds, their $x$- and $y$- positions must both be the same at this time.

\[ x_A = \frac{2}{t} \]  \hspace{1cm} (1)

\[ x_B = 5 - 2t \]  \hspace{1cm} (2)

Set their $x$-positions equal.

\[ \frac{2}{t} = 5 - 2t \]

\[ 2 = 5t - 2t^2 \]

\[ 2t^2 - 5t + 2 = 0 \]

\[ (2t-1)(t-2) = 0 \]

Either $t = \frac{1}{2}$ or $t = 2$

\[ y_A = 3t + 1 \]  \hspace{1cm} (3)

\[ y_B = 2t^2 + 2k - 1 \]  \hspace{1cm} (4)

Set their $y$-positions equal and rearrange for $k$.

\[ 2t^2 + 2k - 1 = 3t + 1 \]

\[ 2k = 2 + 3t - 2t^2 \]

\[ k = \frac{2 + 3t - 2t^2}{2} \]  \hspace{1cm} (5)

Substitute $t = \frac{1}{2}$ into (5).

\[ k = \frac{2 + 3\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^2}{2} = \frac{3}{2} \]

Substitute $t = 2$ into (5).

\[ k = \frac{2 + 3(2) - 2(2)^2}{2} = 0 \]

Since $k > 0$, $k = \frac{3}{2}$

**b** $k = \frac{3}{2}$ when $t = \frac{1}{2}$

Substitute $t = \frac{1}{2}$ into (2) and (3).

\[ x = 5 - 2\left(\frac{1}{2}\right) = 4 \]

\[ y = 3\left(\frac{1}{2}\right) + 1 = \frac{5}{2} \]

The coordinates of the point of collision are \(\left(\frac{1}{2}, \frac{5}{2}\right)\).

Note that you can check your answer by using (1) and (4).