Parametric equations, Mixed exercise 8

1 a At A, $y = 0 \Rightarrow 3\sin t = 0 \Rightarrow \sin t = 0$
So $t = 0$ or $t = \pi$

Substitute $t = 0$ and $t = \pi$ into
$x = 4\cos t$
$t = 0 \Rightarrow x = 4\cos 0 = 4 \times 1 = 4$
$t = \pi \Rightarrow x = 4\cos\pi = 4 \times (-1) = -4$
The coordinates of A are $(4, 0)$.

At B, $x = 0 \Rightarrow 4\cos t = 0 \Rightarrow \cos t = 0$
So $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$
Substitute $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ into
$y = 3\sin t$
$t = \frac{\pi}{2} \Rightarrow y = 3\sin\frac{\pi}{2} = 3 \times 1 = 3$
$t = \frac{3\pi}{2} \Rightarrow y = 3\sin\frac{3\pi}{2} = 3 \times -1 = -3$
The coordinates of B are $(0, 3)$.

b At C, $t = \frac{\pi}{6}$
$x = 4\cos\frac{\pi}{6} = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$y = 3\sin\frac{\pi}{6} = 3 \times \frac{1}{2} = \frac{3}{2}$
The coordinates of C are $\left(2\sqrt{3}, \frac{3}{2}\right)$.

c $x = 4\cos t \Rightarrow \frac{x}{4} = \cos t \qquad (1)$

$y = 3\sin t \Rightarrow \frac{y}{3} = \sin t \qquad (2)$

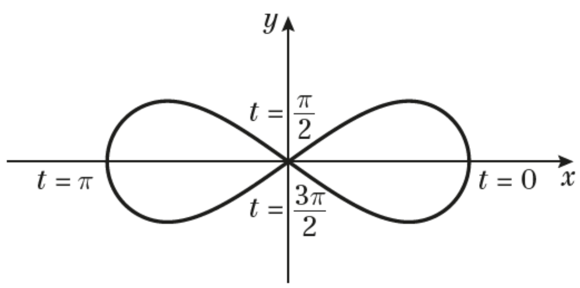
Substitute (1) and (2) into
$\cos^2 t + \sin^2 t = 1$
$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$
$\frac{x^2}{16} + \frac{y^2}{9} = 1$
$9x^2 + 16y^2 = 144$

2 Substitute $t = 0$ into
$x = \cos t$ and $y = \frac{1}{2}\sin 2t$
$x = \cos 0 = 1$
$y = \frac{1}{2}\sin(2 \times 0) = \frac{1}{2}\sin 0 = \frac{1}{2} \times 0 = 0$
So when $t = 0$, $(x, y) = (1, 0)$.

Substitute $t = \frac{\pi}{2}$ into
$x = \cos t$ and $y = \frac{1}{2}\sin 2t$
$x = \cos\frac{\pi}{2} = 0$
$y = \frac{1}{2}\sin\left(2 \times \frac{\pi}{2}\right) = \frac{1}{2}\sin\pi = \frac{1}{2} \times 0 = 0$
So when $t = \frac{\pi}{2}$, $(x, y) = (0, 0)$.

Substitute $t = \pi$ into
$x = \cos t$ and $y = \frac{1}{2}\sin 2t$
$x = \cos\pi = -1$
$y = \frac{1}{2}\sin 2\pi = \frac{1}{2} \times 0 = 0$
So when $t = \pi$, $(x, y) = (-1, 0)$.

Substitute $t = 3\pi$ into
$x = \cos t$ and $y = \frac{1}{2}\sin 2t$
$x = \cos\frac{3\pi}{2} = 0$
$y = \frac{1}{2}\sin\left(2 \times \frac{3\pi}{2}\right) = \frac{1}{2}\sin 3\pi = \frac{1}{2} \times 0 = 0$
So when $t = \frac{3\pi}{2}$, $(x, y) = (0, 0)$.



3 a $x = e^{2t+1} + 1 \quad (1)$

$x - 1 = e^{2t+1}$

$\ln(x-1) = 2t + 1$

$\ln(x-1) - 1 = 2t$

$\frac{1}{2}\ln(x-1) - \frac{1}{2} = t \quad (2)$

$y = t + \ln 2 \quad (3)$

Substitute (2) into (3).

$y = \frac{1}{2}\ln(x-1) - \frac{1}{2} + \ln 2$

$= \ln(x-1)^{\frac{1}{2}} + \ln 2 - \frac{1}{2}$

$= \ln\left(2\sqrt{x-1}\right) - \frac{1}{2}$

The Cartesian equation of the curve is

$y = \ln\left(2\sqrt{x-1}\right) - \frac{1}{2}$

Substitute $t = 1$ into (1).

$x = e^3 + 1$

Since x is an increasing function and $t > 1$,

$x > e^3 + 1$

So $k = e^3 + 1$

b The range of f(x) is the range of $y = q(t)$ so substitute $t = 1$ into (3).

$y = 1 + \ln 2$

Since y is an increasing function and $t > 1$,

$y > 1 + \ln 2$

The range of f(x) is $y > 1 + \ln 2$.

4 $x = \frac{1}{2t+1} \quad (1)$

$2t + 1 = \frac{1}{x}$

$2t = \frac{1}{x} - 1$

$t = \frac{1}{2x} - \frac{1}{2} \quad (2)$

$y = 2\ln\left(t + \frac{1}{2}\right) \quad (3)$

Substitute (2) into (3).

$y = 2\ln\left(\frac{1}{2x} - \frac{1}{2} + \frac{1}{2}\right)$

$= 2\ln\left(\frac{1}{2x}\right)$

$= -2\ln(2x)$

$= -2(\ln 2 + \ln x)$

$= -\ln 4 - 2\ln x$

The Cartesian equation of the curve is

$y = -\ln 4 - 2\ln x$

The domain of f(x) is the range of $x = p(t)$ so substitute $t = \frac{1}{2}$ into (1).

$x = \frac{1}{2\left(\frac{1}{2}\right) + 1} = \frac{1}{2}$

As $t \to \infty$, $x \to 0$.

So the domain is $0 < x < \frac{1}{2}$.

The range of f(x) is the range of $y = q(t)$ so substitute $t = \frac{1}{2}$ into (3).

$y = 2\ln\left(\frac{1}{2} + \frac{1}{2}\right) = 0$

As $t \to \infty$, $y \to \infty$

So the range is $y > 0$.

5 a $x = \sin t$ (1)

$y = \cos^2 t$ (2)

Substitute (1) and (2) into

$\cos^2 t = 1 - 2\sin^2 t$

$y = 1 - 2x^2$

A Cartesian equation of the curve is

$y = 1 - 2x^2$

b Substitute $y = 0$ into

$y = 1 - 2x^2$

$0 = 1 - 2x^2$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{1}}{\sqrt{2}} = \pm\frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

So the curve meets the x-axis at

$\left(\frac{\sqrt{2}}{2}, 0\right)$ and $\left(-\frac{\sqrt{2}}{2}, 0\right)$.

So a and b are $\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2}$.

6 $x = \frac{1}{1+t}$ (1)

$x(1+t) = 1$

$1 + t = \frac{1}{x}$

So $t = \frac{1}{x} - 1$ (2)

Substitute (2) into

$$y = \frac{1}{(1+t)(1-t)}$$

$$y = \frac{1}{\left(1 + \frac{1}{x} - 1\right)\left(1 - \left(\frac{1}{x} - 1\right)\right)}$$

$$= \frac{1}{\left(\frac{1}{x}\right)\left(2 - \frac{1}{x}\right)}$$

$$= \frac{x^2}{x^2\left(\frac{1}{x}\right)\left(2 - \frac{1}{x}\right)}$$

$$= \frac{x^2}{2x - 1}$$

So the Cartesian equation of the curve

is $y = \frac{x^2}{2x - 1}$

7 a $x = 4\sin t - 3 \Rightarrow \sin t = \frac{x+3}{4}$ (1)

$y = 4\cos t + 5 \Rightarrow \cos t = \frac{y-5}{4}$ (2)

Substitute (1) and (2) into

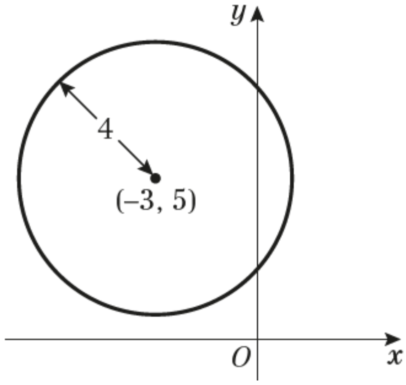
$\sin^2 t + \cos^2 t = 1$

$$\left(\frac{x+3}{4}\right)^2 + \left(\frac{y-5}{4}\right)^2 = 1$$

$$\frac{(x+3)^2}{16} + \frac{(y-5)^2}{16} = 1$$

$$(x+3)^2 + (y-5)^2 = 16$$

7 b The circle $(x+3)^2+(y-5)^2=4^2$ has centre $(-3, 5)$ and radius 4.



c Substitute $x = 0$ into

$$(x+3)^2+(y-5)^2=16$$
$$(0+3)^2+(y-5)^2=16$$
$$3^2+(y-5)^2=16$$
$$9+(y-5)^2=16$$
$$(y-5)^2=7$$
$$y-5=\pm\sqrt{7}$$
$$y=5\pm\sqrt{7}$$

The points of intersectin of the circle and the y-axis are at $\left(0, 5+\sqrt{7}\right)$ and $\left(0, 5-\sqrt{7}\right)$.

8 a $x=\dfrac{2-3t}{1+t}$ (1)

$$x+xt=2-3t$$
$$xt+3t=2-x$$
$$t(x+3)=2-x$$

$t=\dfrac{2-x}{x+3}$ (2)

$y=\dfrac{3+2t}{1+t}$ (3)

Substitute (2) into (3).

$$y=\frac{3+2\left(\frac{2-x}{x+3}\right)}{1+\left(\frac{2-x}{x+3}\right)}$$
$$=\frac{\frac{3(x+3)+2(2-x)}{x+3}}{\frac{x+3+2-x}{x+3}}$$
$$=\frac{3(x+3)+2(2-x)}{x+3+2-x}$$
$$=\frac{3x+9+4-2x}{5}$$
$$=\frac{x+13}{5}$$
$$=\frac{1}{5}x+\frac{13}{5}$$

This is in the form $y = mx + c$, therefore the curve C is a straight line.

8 b Substitute $t = 0$ into (1) and (2).

$$x = \frac{2 - 3(0)}{1 + 0} = 2$$

$$y = \frac{3 + 2(0)}{1 + 0} = 3$$

Coordinates are $(2, 3)$.

Substitute $t = 4$ into (1) and (2).

$$x = \frac{2 - 3(4)}{1 + 4} = -2$$

$$y = \frac{3 + 2(4)}{1 + 4} = \frac{11}{5}$$

Coordinates are $\left(-2, \frac{11}{5}\right)$

$$\text{Length} = \sqrt{(2 - (-2))^2 + \left(3 - \frac{11}{5}\right)^2}$$

$$\sqrt{(4)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{416}{25}}$$

$$= \frac{4\sqrt{26}}{5}$$

9 a $x = t^2 - 2$

$x + 2 = t^2$

$\pm\sqrt{x + 2} = t$

But $0 \le t \le 2$ so choose the positive value.

$t = \sqrt{x + 2}$ (1)

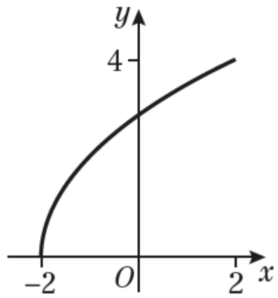
Substitute (1) into

$y = 2t$

$y = 2\sqrt{x + 2}$

b Domain of f(x) is $-2 \le x \le 2$.
Range of f(x) is $0 \le y \le 4$.

c



10 a $x = 2\cos t \Rightarrow \frac{x}{2} = \cos t$ (1)

$y = 2\sin t - 5 \Rightarrow \frac{y + 5}{2} = \sin t$ (2)

Substitute (1) and (2) into

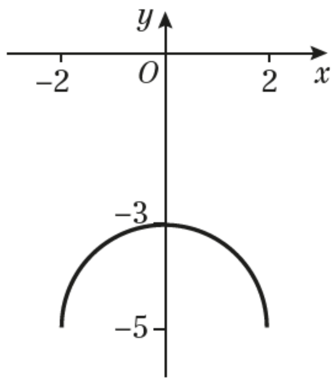
$\cos^2 t + \sin^2 t = 1$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y + 5}{2}\right)^2 = 1$$

$x^2 + (y + 5)^2 = 4$

So the curve C forms part of a circle of radius 2 and centre (0, −5).

b



c Since $0 \le t \le \pi$, the curve C forms half of the circle.

Arc length = $r\theta = 2\pi$

11 a $x = t - 2 \Rightarrow x + 2 = t$ (1)

Substitute (1) into

$y = t^3 - 2t^2$

$y = (x+2)^3 - 2(x+2)^2$

$= (x+2)^2(x+2-2)$

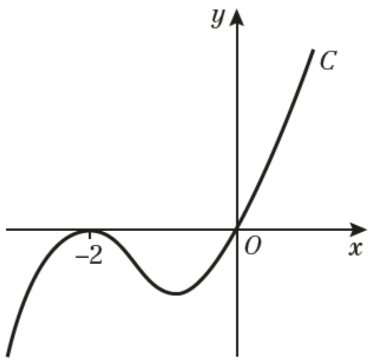
$= x(x^2 + 4x + 4)$

$= x^3 + 4x^2 + 4x$

The Cartesian equation of C is

$y = x^3 + 4x^2 + 4x$

b



12 $x = t - 3$ (1)

$y = 4 - t^2$ (2)

Substitute (1) and (2) into

$y = 4x + 20$

$4 - t^2 = 4(t-3) + 20$

$0 = t^2 + 4t + 4$

Use the discriminant:

'$b^2 - 4ac$' $= 4^2 - 4(1)(4)$

$= 0$

Therefore, the line and the curve have only one point of intersection, that is, they touch. So the line is a tangent to the curve.

13 a $x = 2\ln t$ (1)

$x = 5$ (2)

Substitute (2) into (1).

$5 = 2\ln t$

$\frac{5}{2} = \ln t$

$e^{\frac{5}{2}} = t$ (3)

Substitute (3) into

$y = t^2 - 1$

$y = \left(e^{\frac{5}{2}}\right)^2 - 1$

$= e^5 - 1$

The coordinates of the point of intersection of the line $x = 5$ and the curve are $(5, e^5 - 1)$.

b $y = t^2 - 1$, $t > 0$

(In this domain) the function is increasing. So the range is $y > -1$.

Therefore, $k > -1$.

14 a At A, $x = 0$

$x = 1 + 2t$ (1)

$0 = 1 + 2t$

$-\frac{1}{2} = t$

So substitute $t = -\frac{1}{2}$ into

$y = 4^t - 1$

$y = 4^{-\frac{1}{2}} - 1 = -\frac{1}{2}$

The coordinates of A are $\left(0, -\frac{1}{2}\right)$.

At B, $y = 0$

$y = 4^t - 1$

$0 = 4^t - 1$

$1 = 4^t$

$t = 0$

So substitute $t = 0$ into (1).

$x = 1 + 2t$

$x = 1 + 2(0) = 1$

The coordinates of B are $(1, 0)$.

b $m_l = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0 - \left(-\frac{1}{2}\right)}{1 - 0} = \frac{1}{2}$

Substitute $(1, 0)$ and $m = \frac{1}{2}$ into

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{1}{2}(x - 1)$

$2y = x - 1$

$x - 2y - 1 = 0$

15 At A, $x = 0$

$x = \ln t - \ln\left(\frac{\pi}{2}\right)$ (1)

$0 = \ln t - \ln\left(\frac{\pi}{2}\right)$

$\ln\left(\frac{\pi}{2}\right) = \ln t$

$\frac{\pi}{2} = t$

$y = \sin t$ (2)

Substitute $t = \frac{\pi}{2}$ into (2).

$y = \sin\frac{\pi}{2} = 1$

The coordinates of A are $(0, 1)$.

At B, $y = 0$

$y = \sin t$

$0 = \sin t$, $0 < t < 2\pi$

$\pi = t$

Substitute $t = \pi$ into (1).

$x = \ln\pi - \ln\left(\frac{\pi}{2}\right)$

$= \ln\pi - (\ln\pi - \ln 2) = \ln 2$

The coordinates of B are $(\ln 2, 0)$.

$m_l = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0 - 1}{\ln 2 - 0} = -\frac{1}{\ln 2}$

Substitute $(0, 1)$ and $m = -\frac{1}{\ln 2}$ into

$y - y_1 = m(x - x_1)$

$y - 1 = -\frac{1}{\ln 2}(x - 0)$

$y\ln 2 - \ln 2 = -x$

$x + y\ln 2 - \ln 2 = 0$

16 a $x = 80t$ (1)

$\frac{x}{80} = t$ (2)

$y = 3000 - 30t$ (3)

Substitute (2) into (3).

$y = 3000 - 30\left(\frac{x}{80}\right)$

$= 3000 - \frac{3}{8}x$

This is in the form $y = mx + c$, therefore the plane's descent is a straight line.

b Substitute $y = 30$ into (3).

$30 = 3000 - 30t$
$1 = 100 - t$
$t = 99$

So $k = 99$

c Substitute $t = 99$ into (1).

$x = 80(99) = 7920$

During this portion of its descent the plane travels 7920 m horizontally and descends vertically by 3000 − 30 = 2970 m.

Distance travelled $= \sqrt{7920^2 + 2970^2}$

$= 8458.56$ m (2 d.p.)

17 a Substitute $y = 0$ into

$y = 1.5 - 4.9t^2 + 50\sqrt{2}t$ (1)

$0 = 1.5 - 4.9t^2 + 50\sqrt{2}t$

$4.9t^2 - 50\sqrt{2}t - 1.5 = 0$

$t = \frac{50\sqrt{2} \pm \sqrt{2(50)^2 - 4(4.9)(-1.5)}}{2(4.9)}$

$= 14.45...$ or $t = -0.021...$

$t = 14.45$ (since $t > 0$)

$x = 50\sqrt{2}t$ (2)

Substitute $t = 14.45$ into (2).

$x = 50\sqrt{2}(14.45...)$

$= 1022...$

The furthest horizontal distance is 1022 m (4 s.f.).

b Substitute $x = 1000$ into (2).

$1000 = 50\sqrt{2}t$

$t = \frac{1000}{50\sqrt{2}}$

$= \frac{20}{\sqrt{2}} = 10\sqrt{2}$

Substitute $t = 10\sqrt{2}$ into (1).

$y = 1.5 - 4.9\left(10\sqrt{2}\right)^2 + 50\sqrt{2}\left(10\sqrt{2}\right)$

$= 1.5 - 490 \times 2 + 500 \times 2$

$= 1.5 + 20$

$= 21.5$

$21.5 > 10$, so the arrow is too high to hit the castle wall.

c Substitute $y = 10$ into (1).

$10 = 1.5 - 4.9t^2 + 50\sqrt{2}t$

$4.9t^2 - 50\sqrt{2}t + 8.5 = 0$

$t = \frac{50\sqrt{2} \pm \sqrt{2(50)^2 - 4(4.9)(8.5)}}{2(4.9)}$

$= 14.309...$ (or $t = 0.1212...$)

$x = 50\sqrt{2}(14.309...) = 1011.799...$

The archer needs to move back by 11.8 m (3 s.f.).

18 a $y = 244t(4-t), \quad 0 < t < 4$ (1)

y is a parabola with midpoint (which corresponds to the maximum height) at $t = 2$ hours.

Substitute $t = 2$ into (1).

$y = 244(2)(4-2) = 976$ m

b The mountaineer completes her walk at sea level, when $y = 0$.

$y = 0$ when $t = 4$ (and when $t = 0$).

Substitute $t = 4$ into

$x = 300\sqrt{t}$

$x = 300\sqrt{4} = 600$ m

The horizontal distance is 600 m.

19 a The curve is symmetrical in the x-axis. Its highest point is at the midpoint, at which $t = \pi$.

At $t = \pi$:

$$y = -\cos t$$
$$= -\cos \pi$$
$$= 1$$

The height of the bridge is 10 m.

b At $t = \frac{\pi}{2}$:

$$x = \frac{4t}{\pi} - 2\sin t$$
$$= \frac{4}{\pi} \times \frac{\pi}{2} - 2\sin\frac{\pi}{2}$$
$$= 0$$

At $t = \frac{3\pi}{2}$:

$$x = \frac{4}{\pi} \times \frac{3\pi}{2} - 2\sin\frac{3\pi}{2} = 8$$

The maximum width is 80 – 0 = 80 m.

20 a $y = 10(t-1)^2$ (1)

Substitute $t = 0$ into (1).

$$y = 10(0-1)^2 = 10$$

The cyclist's initial height is 10 m.

b Substitute $y = 0$ into (1).

$$0 = 10(t-1)^2$$
$$0 = (t-1)^2$$
$$t = 1$$

The cyclist is at her lowest height after 1 second.

c Substitute $t = 1.3$ into (1).

$$y = 10(1.3-1)^2 = 0.9$$

The cyclist leaves the ramp at height 0.9 m.

Challenge

a If the particles collide at time t seconds, their x- and y- positions must both be the same at this time.

$x_A = \frac{2}{t}$ (1)

$x_B = 5 - 2t$ (2)

Set their x-positions equal.

$\frac{2}{t} = 5 - 2t$

$2 = 5t - 2t^2$

$2t^2 - 5t + 2 = 0$

$(2t-1)(t-2) = 0$

Either $t = \frac{1}{2}$ or $t = 2$

$y_A = 3t + 1$ (3)

$y_B = 2t^2 + 2k - 1$ (4)

Set their y-positions equal and rearrange for k.

$2t^2 + 2k - 1 = 3t + 1$

$2k = 2 + 3t - 2t^2$

$k = \frac{2 + 3t - 2t^2}{2}$ (5)

Substitute $t = \frac{1}{2}$ into (5).

$$k = \frac{2 + 3\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^2}{2} = \frac{3}{2}$$

Substitute $t = 2$ into (5).

$$k = \frac{2 + 3(2) - 2(2)^2}{2} = 0$$

Since $k > 0$, $k = \frac{3}{2}$

b $k = \frac{3}{2}$ when $t = \frac{1}{2}$

Substitute $t = \frac{1}{2}$ into (2) and (3).

$x = 5 - 2\left(\frac{1}{2}\right) = 4$

$y = 3\left(\frac{1}{2}\right) + 1 = \frac{5}{2}$

The coordinates of the point of collision are $\left(4, \frac{5}{2}\right)$.

Note that you can check your answer by using (1) and (4).