## Parametric equations 8E

**1 a** Substitute $x = 75$ into

$x = 0.9t$

$75 = 0.9t$

$t = \dfrac{10 \times 75}{9}$

$= \dfrac{250}{3} = 83.3\,\text{seconds}$

**b** $y = -3.2t$

$y = -3.2\left(\dfrac{250}{3}\right)$

$= -\dfrac{800}{3} = -267$ (3 s.f.)

The boat has been moved off course by 267 m southwards.

**c** $x = 0.9t \Rightarrow \dfrac{10}{9}x = t \qquad (1)$

Substitute (1) into

$y = -3.2t$

$y = -3.2\left(\dfrac{10}{9}x\right)$

$= -\dfrac{32}{9}x$

This equation is in the form $y = mx\ (+\ c)$ and so is a straight line.

**d** Total distance travelled is

$\sqrt{75^2 + \left(\dfrac{800}{3}\right)^2} = \dfrac{\sqrt{690625}}{3}$

$= 277\,\text{m}$ (3 s.f.)

$\text{Speed} = \dfrac{\text{distance}}{\text{time}}$

$= \dfrac{\frac{\sqrt{690625}}{3}}{\frac{250}{3}} = \dfrac{\sqrt{690625}}{250}$

$= 3.324...$

The speed is $3.32\,\text{m s}^{-1}$ (3 s.f.).

**2 a** Substitute $t = 0$ into

$y = -9.1t + 3000$

$y = -9.1(0) + 3000$

$= 3000$

The initial height is 3000 m.

**b** Substitute $y = 0$ into

$y = -9.1t + 3000$

$0 = -9.1t + 3000$

$9.1t = 3000$

$t = \dfrac{3000}{9.1} = 329.67\,\text{minutes}$

$t = 0$ is the start of the descent.

When $t \geq 300$, $y < 0$, so the plane would be underground or below sea level.

**c** Substitute $t = \dfrac{30\,000}{91}$ into

$x = 80t$

$x = 80\left(\dfrac{30\,000}{91}\right) = 26\,373$

The horizontal distance travelled during the descent is 26 400 m (3 s.f.).

**3** $x = 10\sqrt{3}t\ \text{m} \qquad (1)$

$y = (-4.9t^2 + 10t)\ \text{m} \qquad (2)$

$0 \leq t \leq k$ seconds

**a** Substitute $y = 0$ into (2).

$y = -4.9t^2 + 10t$

$0 = -t(4.9t - 10)$

Either $t = 0$ (when the ball is kicked) or

$t = \dfrac{100}{49} = 2.04$

Substitute $t = \dfrac{100}{49}$ into (1).

$x = 10\sqrt{3}t$

$= 10\sqrt{3}\left(\dfrac{100}{49}\right) = 35.34...$

The distance is 35.3 m (3 s.f.).

**3 b** Substitute $y = 1.5$ into (2).

$y = -4.9t^2 + 10t$

$1.5 = -4.9t^2 + 10t$

$4.9t^2 - 10t + 1.5 = 0$

$t = \dfrac{10 \pm \sqrt{100 - 4(4.9)(1.5)}}{2(4.9)}$

Either $t = 1.88$ or $t = 0.163$

On the descent, $t = 1.88$ seconds (3 s.f.).

Substitute $y = 2.5$ into (2).

$4.9t^2 - 10t + 2.5 = 0$

$t = \dfrac{10 \pm \sqrt{100 - 4(4.9)(2.5)}}{2(4.9)}$

Either $t = 1.75$ or $t = 0.292$

On the descent, $t = 1.75$ seconds (3 s.f.).

The range of time is $1.75 \le t \le 1.88$ seconds (3 s.f.).

**c** The closest distance is at the lower end of the range of times, so substitute $t = 1.75$ into (1).

$x = 10\sqrt{3}t$

$x = 10\sqrt{3} \times 1.75 = 30.3$

The closest distance is 30.3 m (3 s.f.)

**4** $x = 2t$ m (1)

$y = (-4.9t^2 + 10t)$ m (2)

**a** Substitute $y = 0$ into (2).

$y = -4.9t^2 + 10t$

$0 = -t(4.9t - 10)$

Either $t = 0$ (at the start) or $t = \dfrac{100}{49}$

The time taken is $t = \dfrac{100}{49}$ seconds.

**b** Substitute $t = \dfrac{100}{49}$ into (1).

$x = 2t$

$x = 2\left(\dfrac{100}{49}\right) = \dfrac{200}{49}$ m

**c** $x = 2t \Rightarrow t = \dfrac{x}{2}$ (3)

Substitute (3) into (2).

$y = -4.9\left(\dfrac{x}{2}\right)^2 + 10\left(\dfrac{x}{2}\right)$

$= -\dfrac{4.9}{4}x^2 + 5x$

$= -\dfrac{49}{40}x^2 + 5x$ (4)

Therefore, the path is a quadratic curve.

**d** For (maximum) height use either (2) or (4).

From (2):

$y = -4.9t^2 + 10t$

$\dfrac{dy}{dt} = -9.8t + 10$

$0 = -9.8t + 10$

$t = \dfrac{10}{9.8} = \dfrac{100}{98}$ seconds

Substitute $t = \dfrac{100}{98}$ into (2).

$y = -4.9\left(\dfrac{100}{98}\right)^2 + 10\left(\dfrac{100}{98}\right) = \dfrac{250}{49}$

The maximum height is $\dfrac{250}{49}$ m.

**5** $x = 12\sin t$ (1)
$y = 12 - 12\cos t$ (2)
$t \geq 0$ minutes

**a** Rearrange (1) and (2).

$x = 12\sin t \Rightarrow \frac{x}{12} = \sin t$ (3)

$y = 12 - 12\cos t \Rightarrow \cos t = \frac{12-y}{12}$ (4)

Substitute (3) and (4) into
$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x}{12}\right)^2 + \left(\frac{12-y}{12}\right)^2 = 1$

$x^2 + (12-y)^2 = 12^2$

or

$x^2 + (-(y-12))^2 = 12^2$

$x^2 + (-1)^2(y-12)^2 = 12^2$

$x^2 + (y-12)^2 = 12^2$

The motion is a circle, centre (0, 12) and radius 12 m.

**b** The car is at its maximum height when $t = \pi$ minutes.

Substitute $t = \pi$ into (2).
$y = 12 - 12\cos\pi = 24$

The maximum height is 24 m.

**c** Time ($t$) = $2\pi$ minutes

Distance ($d$) = $2\pi r$
$= 2\pi \times 12$
$= 24\pi$ m

Average speed = $\frac{d}{t} = \frac{24\pi}{2\pi} = 12$

The average speed is 12 m $\text{min}^{-1}$.

**6** $x = t - 4\sin t$ (1)
$y = 1 - 2\cos t$ (2)
$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

**a** Substitute $t = -\frac{\pi}{2}$ into (1).

$x = t - 4\sin t$

$x = -\frac{\pi}{2} - 4\sin\left(-\frac{\pi}{2}\right) = 4 - \frac{\pi}{2}$

Substitute $t = \frac{\pi}{2}$ into (1).

$x = \frac{\pi}{2} - 4\sin\frac{\pi}{2} = \frac{\pi}{2} - 4$

Length of opening
$= 4 - \frac{\pi}{2} - \left(\frac{\pi}{2} - 4\right)$
$= 8 - \pi = 4.86$ (3 s.f.)

**b** Substitute $t = \frac{\pi}{2}$ into (2).

$y = 1 - 2\cos t$

$y = 1 - 2\cos\left(\frac{\pi}{2}\right) = 1$

$y$ is a minimum when $t = 0$ (by symmetry) so substitute $t = 0$ into (2).
$y = 1 - 2\cos(0) = -1$

Depth $= 1 - (-1) = 2$

**7** $x = \frac{t^2 - 3t + 2}{t}$ (1)
$y = 2t$ (2)
$t > 0$ seconds

**a** Substitute $t = 0.5$ into (1) and then (2).

$x = \frac{(0.5)^2 - 3(0.5) + 2}{0.5} = 1.5$

$y = 2 \times 0.5 = 1$

Distance from origin
$= \sqrt{\left(\frac{3}{2}\right)^2 + 1^2} = \frac{\sqrt{13}}{2}$

**7 b** Substitute $x = 0$ into (1).

$x = \dfrac{t^2 - 3t + 2}{t}$

$0 = \dfrac{t^2 - 3t + 2}{t}$

$= t^2 - 3t + 2$

$= (t-1)(t-2)$

Either $t = 1$ or $t = 2$

Substitute $t = 1$ into (2).

$y = 2t$

$y = 2(1) = 2$

Substitute $t = 2$ into (2).

$y = 2t$

$y = 2(2) = 4$

Coordinates are $(0, 2)$ and $(0, 4)$.

**c** Substitute (1) and (2) into

$y = 2x + 10$

$2t = 2\left(\dfrac{t^2 - 3t + 2}{t}\right) + 10$

$2t^2 = 2(t^2 - 3t + 2) + 10t$

$2t^2 = 2t^2 - 6t + 4 + 10t$

$-4 = 4t$

$-1 = t$

But $t > 0$, therefore the two particles never meet.

**8** $x = 18t$ (1)

$y = -4.9t^2 + 4t + 10$ (2)

$0 \le t \le k$

**a** Substitute $t = 0$ into (2).

$y = -4.9(0)^2 + 4(0) + 10 = 10$

The initial height is 10 m.

**b** Substitute $y = 0$ into (2).

$y = -4.9t^2 + 4t + 10$

$0 = -4.9t^2 + 4t + 10$

$0 = 4.9t^2 - 4t - 10$

$t = \dfrac{4 \pm \sqrt{16 - 4(4.9)(-10)}}{2(4.9)}$

$= \dfrac{4 \pm \sqrt{212}}{9.8}$

Either $t = 1.89...$ or $t = -1.08...$

But $t \ge 0$ (and $t \le k$), so $k = 1.89$ (3 s.f.).

The time taken is 1.89 seconds.

**c** Substitute $t = 1.89...$ into (1).

$x = 18t$

$x = 18\left(\dfrac{4 + \sqrt{212}}{9.8}\right) = 34.1$

The horizontal distance travelled is 34.1 m (3 s.f.).

**8 d** Rearrange (1).

$x = 18t \Rightarrow t = \dfrac{x}{18}$ (3)

Substitute (3) into (2).

$$y = -4.9\left(\frac{x}{18}\right)^2 + 4\left(\frac{x}{18}\right) + 10$$
$$= -\frac{4.9}{324}x^2 + \frac{4}{18}x + 10$$
$$= -\frac{49}{3240}x^2 + \frac{2}{9}x + 10 \quad (4)$$

Since (4) is a quadratic equation, the ski jumper's path is a parabola.

For (maximum) height use (2) or (4).

Choosing (2), find $\dfrac{dy}{dt}$.

$$y = -4.9t^2 + 4t + 10$$
$$\frac{dy}{dt} = -9.8t + 4$$
$$0 = -9.8t + 4$$
$$9.8t = 4$$
$$t = \frac{4}{9.8} = \frac{20}{49}$$

Substitute $t = \dfrac{20}{49}$ into (2).

$$y = -4.9t^2 + 4t + 10$$
$$y = -4.9\left(\frac{20}{49}\right)^2 + 4\left(\frac{20}{49}\right) + 10$$
$$= \frac{530}{49} = 10.81\ldots$$

The maximum height above ground is 10.8 m (3 s.f.).

**9** $x = 50\tan t$ m (1)
$y = 20\sin 2t$ m (2)

$0 < t \le \dfrac{\pi}{2}$

**a** Maximum value of $y$ is when

$$\sin 2t = 1$$
$$2t = \frac{\pi}{2}$$
$$t = \frac{\pi}{4}$$

**b** Substitute $t = \dfrac{\pi}{4}$ into (1) and (2).

$$x = 50\tan\left(\frac{\pi}{4}\right) = 50 \text{ m}$$
$$y = 20\sin 2\left(\frac{\pi}{4}\right) = 20 \text{ m}$$

Coordinates are $(50, 20)$.

**c** Substitute $t = 1$ into (1) and (2).

$x = 50\tan 1 = 77.87\ldots$ m
$y = 20\sin 2 = 18.18\ldots$ m

Coordinates are $(77.9, 18.2)$ (3 s.f.).

Note that $\dfrac{\pi}{4} < 1 < \dfrac{\pi}{2}$, and that the sine function is continuous between $\sin 2\left(\dfrac{\pi}{4}\right) = 1$ and $\sin 2\left(\dfrac{\pi}{2}\right) = 0$.

So, $\sin 2\left(\dfrac{\pi}{4}\right) > \sin 2(1) > \sin 2\left(\dfrac{\pi}{2}\right)$

or $1 > \sin 2(1) > 0$.

Therefore, around $t = 1$, as $t$ increases, $y$ decreases. The cyclist is descending.

**10** $x = 5 + \ln t$ (1)
$y = 5\sin 2t$ (2)

$0 \le t \le \dfrac{\pi}{2}$

**a** Substitute $t = \dfrac{\pi}{6}$ into (1) and (2).

$$x = 5 + \ln\frac{\pi}{6} = 4.35 \text{ (3 s.f.)}$$
$$y = 5\sin 2\left(\frac{\pi}{6}\right)$$
$$= \frac{5\sqrt{3}}{2} = 4.33 \text{ (3 s.f.)}$$

Coordinates are $(4.35, 4.33)$.

**10 b** Maximum value of $y$ occurs when
$\sin 2t = 1$

Substitute $\sin 2t = 1$ into (2).
$y = 5\sin 2t$
$y = 5(1) = 5$

Maximum height $= 5 \times 5 = 25$ m

**c** Find $t$ when $y$ is a maximum.
$\sin 2t = 1$
$2t = \frac{\pi}{2}$ (only one solution to consider)
$t = \frac{\pi}{4}$

When $t = \frac{\pi}{4}$ (at maximum height):
$x = 5 + \ln t = 5 + \ln\frac{\pi}{4}$

When $t = \frac{\pi}{2}$ (at end of descent):
$x = 5 + \ln\frac{\pi}{2}$

Horizontal distance covered
$= 5 + \ln\frac{\pi}{2} - \left(5 + \ln\frac{\pi}{4}\right)$
$= \ln\frac{\pi}{2} - \ln\frac{\pi}{4}$
$= \ln\left(\frac{\frac{\pi}{2}}{\frac{\pi}{4}}\right) = \ln 2$

Distance $= 5\ln 2 = 3.47$ m (3 s.f.).

**d** Gradient $= \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{0 - 25}{5\ln 2}$
$= \frac{-5}{\ln 2}$
$= -7.21$ (3 s.f.)