Parametric equations 8D

1 a The curve meets the x-axis when $y = 0$

$y = 6 - t$

$0 = 6 - t$

So $t = 6$

Substitute $t = 6$ into the parametric equation for x:

$x = 5 + t$

$x = 5 + 6 = 11$

The coordinates are (11, 0).

b The curve meets the x-axis when $y = 0$

$y = 2t - 6$

$0 = 2t - 6$

$6 = 2t$

So $t = 3$

Substitute $t = 3$ into the parametric equation for x:

$x = 2t + 1$

$x = 2 \times 3 + 1 = 7$

The coordinates are (7, 0).

c The curve meets the x-axis when $y = 0$

$y = (1 - t)(t + 3)$

$0 = (1 - t)(t + 3)$

So $t = 1$ or $t = -3$

Substitute each value of t into the parametric equation for x:

$x = t^2$

When $t = 1,\ x = 1^2 = 1$

When $t = -3,\ x = (-3)^2 = 9$

The coordinates are (1, 0) and (9, 0).

d The curve meets the x-axis when $y = 0$

$y = (t - 1)(2t - 1)$

$0 = (t - 1)(2t - 1)$

So $t = 1$ or $t = \frac{1}{2}$

Substitute each value of t into the parametric equation for x:

$x = \frac{1}{t}$

When $t = 1,\ x = \frac{1}{1} = 1$

When $t = \frac{1}{2},\ x = \frac{1}{\frac{1}{2}} = 2$

The coordinates are (1, 0) and (2, 0).

e The curve meets the x-axis when $y = 0$

$y = t - 9$

$t - 9 = 0$

So $t = 9$

Substitute $t = 9$ into the parametric equation for x:

$x = \frac{2t}{1 + t}$

$x = \frac{2(9)}{1 + (9)} = \frac{18}{10} = \frac{9}{5}$

The coordinates are $\left(\frac{9}{5}, 0\right)$.

2 a The curve meets the y-axis when $x = 0$

$x = 2t$

$0 = 2t$

So $t = 0$

Substitute $t = 0$ into the parametric equation for y:

$y = t^2 - 5$

$y = 0^2 - 5 = -5$

The coordinates are (0, −5).

b The curve meets the y-axis when $x = 0$

$x = 3t - 4$

$0 = 3t - 4$

$3t = 4$

So $t = \frac{4}{3}$

Substitute $t = \frac{4}{3}$ into the parametric equation for y:

$y = \frac{1}{t^2} = \left(\frac{1}{t}\right)^2$

$y = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

The coordinates are $\left(0, \frac{9}{16}\right)$.

2 c The curve meets the y-axis when $x = 0$

$x = t^2 + 2t - 3$

$0 = t^2 + 2t - 3$

$0 = (t-1)(t+3)$

So $t = 1$ or $t = -3$

Substitute each value of t into the parametric equation for y:

$y = t(t-1)$

When $t = 1$, $y = 1(1-1) = 0$

When $t = -3$, $y = -3(-3-1) = 12$

The coordinates are (0, 0) and (0, 12).

d The curve meets the y-axis when $x = 0$

$x = 27 - t^3$

$0 = 27 - t^3$

$t^3 = 27$

So $t = 3$

Substitute $t = 3$ into the parametric equation for y:

$y = \dfrac{1}{t-1}$

$y = \dfrac{1}{3-1} = \dfrac{1}{2}$

The coordinates are $\left(0, \dfrac{1}{2}\right)$.

e The curve meets the y-axis when $x = 0$

$x = \dfrac{t-1}{t+1}$

$0 = \dfrac{t-1}{t+1}$

$0 = t - 1$

So $t = 1$

Substitute $t = 1$ into the parametric equation for y:

$y = \dfrac{2t}{t^2+1}$

$y = \dfrac{2\times 1}{1^2+1} = 1$

The coordinates are (0, 1).

3 At point (4, 0), $x = 4$ and $y = 0$

Hence

$4at^2 = 4 \qquad (1)$

$a(2t-1) = 0 \qquad (2)$

Solving equation (2) for t:

$2t - 1 = 0$

$2t = 1$

$t = \dfrac{1}{2}$

Substitute into equation (1):

$4a\left(\dfrac{1}{2}\right)^2 = 4$

$4a \times \dfrac{1}{4} = 4$

$a = 4$

So the value of a is 4.

4 At point (0, −5), $x = 0$ and $y = -5$

Hence

$b(2t-3) = 0 \qquad (1)$

$b(1-t^2) = -5 \qquad (2)$

Solving equation (1) for t:

$2t - 3 = 0$

$2t = 3$

$t = \dfrac{3}{2}$

Substitute into equation (2):

$b\left(1 - \left(\dfrac{3}{2}\right)^2\right) = -5$

$b\left(1 - \dfrac{9}{4}\right) = -5$

$b\left(-\dfrac{5}{4}\right) = -5$

$b = 4$

So the value of b is 4.

5 Substitute $x = 3t + 2$ and $y = 1 - t$ into $y + x = 2$:

$$(1-t)+(3t+2)=2$$
$$1-t+3t+2=2$$
$$2t+3=2$$
$$2t=-1$$
$$t=-\frac{1}{2}$$

Substitute $t=-\frac{1}{2}$ into the parametric equations:

$$x=3t+2=3\left(-\frac{1}{2}\right)+2=-\frac{3}{2}+2=\frac{1}{2}$$
$$y=1-t=1-\left(-\frac{1}{2}\right)=1+\frac{1}{2}=\frac{3}{2}$$

The coordinates of the point of intersection are $\left(\frac{1}{2},\frac{3}{2}\right)$.

6 Substitute $x = t^2$ and $y = 2t$ into $4x - 2y - 15 = 0$:

$$4(t^2)-2(2t)-15=0$$
$$4t^2-4t-15=0$$
$$(2t+3)(2t-5)=0$$

So $2t+3=0$ or $2t-5=0$

$$t=-\frac{3}{2} \text{ or } t=\frac{5}{2}$$

Substitute $t=-\frac{3}{2}$ into the parametric equations:

$$x=t^2=\left(-\frac{3}{2}\right)^2=\frac{9}{4}$$
$$y=2t=2\left(-\frac{3}{2}\right)=-3$$

Substitute $t=\frac{5}{2}$ into the parametric equations:

$$x=t^2=\left(\frac{5}{2}\right)^2=\frac{25}{4}$$
$$y=2t=2\left(\frac{5}{2}\right)=5$$

The coordinates of the points of intersection are $\left(\frac{9}{4},-3\right)$ and $\left(\frac{25}{4},5\right)$.

7 Substitute $x = t^2$ and $y = 2t$ into $x^2 + y^2 - 9x + 4 = 0$:

$$(t^2)^2+(2t)^2-9(t^2)+4=0$$
$$t^4+4t^2-9t^2+4=0$$
$$t^4-5t^2+4=0$$
$$(t^2-4)(t^2-1)=0$$

So $t^2-4=0$ or $t^2-1=0$

$t^2=4$ or $t^2=1$

$t=\pm2$ or $t=\pm1$

Substitute $t=\pm2$ into the parametric equations:

$$x=(\pm2)^2=4$$
$$y=2\times(\pm2)=\pm4$$

Substitute $t=\pm1$ into the parametric equations:

$$x=(\pm1)^2=1$$
$$y=2\times(\pm1)=\pm2$$

The coordinates of the points of intersection are (4, 4), (4, −4), (1, 2) and (1, −2).

8 a The curve meets the x-axis when $y = 0$

$$y=\cos t,\ 0<t<\pi$$
$$0=\cos t$$
$$t=\frac{\pi}{2}$$

Substitute $t=\frac{\pi}{2}$ into the parametric equation for x:

$$x=t^2-1$$
$$x=\left(\frac{\pi}{2}\right)^2-1=\frac{\pi^2}{4}-1$$

Coordinates on the x-axis are $\left(\frac{\pi^2}{4}-1, 0\right)$.

The curve meets the y-axis when $x = 0$

$$x=t^2-1,\ 0<t<\pi$$
$$0=t^2-1$$
$$1=t^2$$

$t=1$ (as $t=-1$ is outside the domain of t)

Substitute $t = 1$ into the parametric equation for y:

$$y=\cos t$$
$$y=\cos 1$$

Coordinates on the y-axis are $(0, \cos 1)$.

8 b The curve meets the x-axis when $y = 0$

$y = 2\cos t + 1,\ \pi < t < 2\pi$

$0 = 2\cos t + 1$

$\cos t = -\frac{1}{2}$

$t = \frac{4\pi}{3}$

Substitute $t = \frac{4\pi}{3}$ into the parametric equation for x:

$x = \sin 2t$

$x = \sin\left(2\left(\frac{4\pi}{3}\right)\right) = \sin\frac{8\pi}{3} = \frac{\sqrt{3}}{2}$

Coordinates on the x-axis are $\left(\frac{\sqrt{3}}{2}, 0\right)$.

The curve meets the y-axis when $x = 0$

$x = \sin 2t,\ \pi < t < 2\pi$

$0 = \sin 2t,\ 2\pi < 2t < 4\pi$

$\therefore 2t = 3\pi$

$t = \frac{3\pi}{2}$

Substitute $t = \frac{3\pi}{2}$ into the parametric equation for y:

$y = 2\cos t + 1$

$y = 2\cos\left(\frac{3\pi}{2}\right) + 1 = 1$

Coordinates on the y-axis are (0, 1).

c The curve meets the x-axis when $y = 0$

$y = \sin t - \cos t,\ 0 < t < \frac{\pi}{2}$

$0 = \sin t - \cos t$

$\cos t = \sin t$

$\tan t = 1$

$t = \frac{\pi}{4}$

Substitute $t = \frac{\pi}{4}$ into the parametric equation for x:

$x = \tan t$

$x = \tan\frac{\pi}{4} = 1$

Coordinates on the x-axis are (1, 0).

The curve meets the y-axis when $x = 0$

$0 = \tan t$

There are no solutions in the domain of t.

So the curve does not meet the y-axis in the given domain of t.

9 a The curve meets the x-axis when $y = 0$

$y = \ln t$

$0 = \ln t$

$t = e^0 = 1$

Substitute $t = 1$ into the parametric equation for x:

$x = e^t + 5$

$x = e^1 + 5 = e + 5$

Coordinates on the x-axis are (e + 5, 0).

The curve meets the y-axis when $x = 0$

$0 = e^t + 5$

$e^t = -5$

This equation has no solutions since $e^t > 0$ always.

So the curve does not meet the y-axis.

9 b The curve meets the x-axis when $y = 0$

$y = t^2 - 64$
$0 = t^2 - 64$
$t^2 = 64$
$t = 8$ (since $t > 0$)

Substitute $t = 8$ into the parametric equation for x:

$x = \ln t$
$x = \ln 8$

Coordinates on the x-axis are $(\ln 8, 0)$.

The curve meets the y-axis when $x = 0$

$x = \ln t$
$0 = \ln t$
$t = e^0 = 1$

Substitute $t = 1$ into the parametric equation for y:

$y = t^2 - 64$
$y = 1^2 - 64 = -63$

Coordinates on the y-axis are $(0, -63)$.

c The curve meets the x-axis when $y = 0$

$y = 2e^t - 1$
$0 = 2e^t - 1$
$1 = 2e^t$
$e^t = \frac{1}{2}$

Substitute $e^t = \frac{1}{2}$ into the parametric equation for x:

$x = e^{2t} + 1 = (e^t)^2 + 1$
$x = \left(\frac{1}{2}\right)^2 + 1 = \frac{5}{4}$

Coordinates on the x-axis are $\left(\frac{5}{4}, 0\right)$.

The curve meets the y-axis when $x = 0$

$0 = e^{2t} + 1$
$e^{2t} = -1$

This equation has no solutions since $e^{2t} > 0$ always.

So the curve does not meet the y-axis.

10 Substitute $x = t^2$ and $y = t$ into $y = -3x + 2$:

$t = -3t^2 + 2$
$3t^2 + t - 2 = 0$
$(3t - 2)(t + 1) = 0$

So $t = \frac{2}{3}$ or $t = -1$

Substitute $t = \frac{2}{3}$ into the parametric equations:

$x = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$
$y = \frac{2}{3}$

Substitute $t = -1$ into the parametric equations:

$x = (-1)^2 = 1$
$y = -1$

The coordinates of the points of intersection are $\left(\frac{4}{9}, \frac{2}{3}\right)$ and $(1, -1)$.

11 Substitute $x = \ln(t-1)$ and $y = \ln(2t-5)$ into $y = x - \ln 3$:

$\ln(2t-5) = \ln(t-1) - \ln 3$
$\ln(2t-5) = \ln\left(\frac{t-1}{3}\right)$
$2t - 5 = \frac{t-1}{3}$
$6t - 15 = t - 1$
$5t = 14$

so $t = \frac{14}{5}$

Substitute $t = \frac{14}{5}$ into the parametric equations:

$x = \ln\left(\frac{14}{5} - 1\right) = \ln\left(\frac{9}{5}\right)$
$y = \ln\left(\frac{28}{5} - 5\right) = \ln\left(\frac{3}{5}\right)$

The coordinates of the points of intersection are $\left(\ln\frac{9}{5}, \ln\frac{3}{5}\right)$.

12 a The curve intersects the x-axis when $y = 0$

$y = 4\sin 2t + 2,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$

$0 = 4\sin 2t + 2$

$\sin 2t = -\frac{1}{2}$

$2t = -\frac{5\pi}{6}, -\frac{\pi}{6}$

$t = -\frac{5\pi}{12}, -\frac{\pi}{12}$

Substitute each value of t into the parametric equation $x = 6\cos t$:

$x = 6\cos\left(-\frac{5\pi}{12}\right) = 6\cos\left(\frac{5\pi}{12}\right)$

$x = 6\cos\left(-\frac{\pi}{12}\right) = 6\cos\left(\frac{\pi}{12}\right)$

The coordinates are

$\left(6\cos\left(\frac{\pi}{12}\right), 0\right)$ and $\left(6\cos\left(\frac{5\pi}{12}\right), 0\right)$.

12 b Substitute the parametric equation $y = 4\sin 2t + 2$ into $y = 4$:

$4\sin 2t + 2 = 4$

$4\sin 2t = 2$

$\sin 2t = \frac{1}{2}$

$2t = \frac{\pi}{6}, \frac{5\pi}{6}$

$t = \frac{\pi}{12}, \frac{5\pi}{12}$ (in the domain $-\frac{\pi}{2} < t < \frac{\pi}{2}$)

c Substitute each value of t into the parametric equations:

When $t = \frac{\pi}{12}$,

$x = 6\cos\frac{\pi}{12}$

$y = 4\sin\left(2\left(\frac{\pi}{12}\right)\right) + 2 = 4\times\frac{1}{2} + 2 = 4$

When $t = \frac{5\pi}{12}$,

$x = 6\cos\frac{5\pi}{12}$

$y = 4\sin\left(2\left(\frac{5\pi}{12}\right)\right) + 2 = 4\times\frac{1}{2} + 2 = 4$.

Coordinates are

$\left(6\cos\frac{\pi}{12}, 4\right)$ and $\left(6\cos\frac{5\pi}{12}, 4\right)$.

13 To find any intersections between the line and the curve, substitute the parametric equations $x = 2t$ and $y = 4t(t-1)$ into $y = 2x - 5$:

$4t(t-1) = 2(2t) - 5$

$4t^2 - 4t = 4t - 5$

$4t^2 - 8t + 5 = 0$

The discriminant of this quadratic equation is

$b^2 - 4ac = (-8)^2 - 4\times 4\times 5$

$= 64 - 80 = -16 < 0$

So the quadratic equation has no real roots. Hence the line does not intersect the curve.

14 a To find intersections between the line and the curve, substitute the parametric equation $y = \cos 2t + 1$ into $y = k$:

$\cos 2t + 1 = k$

$\cos 2t = k - 1$

Since $-1 \le \cos 2t \le 1$,

$-1 \le k - 1 \le 1$

so $0 \le k \le 2$

b First find a Cartesian equation for the curve:

$y = \cos 2t + 1$

$= (1 - 2\sin^2 t) + 1$

$= 2 - 2\sin^2 t$

Since $x = \sin t$,

$y = 2 - 2x^2$

Substitute $y = k$ into this Cartesian equation:

$k = 2 - 2x^2$

$2x^2 + (k-2) = 0$

If $y = k$ is a tangent to the curve, then it touches the curve at one point, so the discriminant of the quadratic is 0.

$b^2 - 4ac = 0$

$0^2 - 4\times 2\times(k-2) = 0$

$-8(k-2) = 0$

$k - 2 = 0$

$\therefore k = 2$

15 a At the point A, $t = \ln 2$

$x = e^{2t} = e^{2\ln 2} = e^{\ln 2^2} = 2^2 = 4$

$y = e^t - 1 = e^{\ln 2} - 1 = 2 - 1 = 1$

$\therefore$ coordinates of A are (4, 1).

At the point B, $t = \ln 3$

$x = e^{2t} = e^{2\ln 3} = e^{\ln 3^2} = 3^2 = 9$

$y = e^t - 1 = e^{\ln 3} - 1 = 3 - 1 = 2$

$\therefore$ coordinates of B are (9, 2).

b Points A and B lie on the line l, so the gradient of l can be found from the coordinates of A and B:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{9-4} = \frac{1}{5}$

15 c Using the gradient and the coordinates of A, the equation of l is given by

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{1}{5}(x - 4)$

$5y - 5 = x - 4$

$x - 5y + 1 = 0$

$x - 5y + 1 = 0$

16 At the point A, $t = \frac{\pi}{6}$

$x = \sin t = \sin\frac{\pi}{6} = \frac{1}{2}$

$y = \cos t = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Coordinates of A are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

At the point B, $t = \frac{\pi}{2}$

$x = \sin t = \sin\frac{\pi}{2} = 1$

$y = \cos t = \cos\frac{\pi}{2} = 0$

Coordinates of B are (1, 0).

As the line l passes through A and B, the gradient of l can be found from the coordinates of A and B.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - \frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

So, using the coordinates of B, the equation of l is given by

$y - y_1 = m(x - x_1)$

$y - 0 = -\sqrt{3}(x - 1)$

$y = -\sqrt{3}x + \sqrt{3}$

$\sqrt{3}x + y - \sqrt{3} = 0$

17 a At A (on the y-axis), $x = 0$

$x = \frac{t-1}{t}$

$0 = \frac{t-1}{t}$

$0 = t - 1$

So $t = 1$

Substitute $t = 1$ into the parametric equation for y:

$y = t - 4$

$y = 1 - 4 = -3$

Coordinates of A are (0, −3).

At B (on the x-axis), $y = 0$

$y = t - 4$

$0 = t - 4$

$t = 4$

Substitute $t = 4$ into the parametric equation for x:

$x = \frac{t-1}{t}$

$x = \frac{4-1}{4} = \frac{3}{4}$

Coordinates of B are $\left(\frac{3}{4}, 0\right)$.

17 b Find the gradient of l_1 using the coordinates of A and B:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-3)}{\frac{3}{4}-0} = \frac{3}{\frac{3}{4}} = 4$$

Because the lines l_2 and l_3 are parallel to l_1, they have the same gradient as l_1 and so have equations

l_1: $y = 4x + c_2$

l_2: $y = 4x + c_3$

Substitute the parametric equations for x and y into $y = 4x + c$:

$$t - 4 = 4\left(\frac{t-1}{t}\right) + c$$

$$t^2 - 4t = 4t - 4 + ct$$

$$t^2 - (8+c)t + 4 = 0 \qquad (1)$$

The lines l_2 and l_3 are tangents to the curve when the discriminant of (1) equals 0.

$$(8+c)^2 - 4 \times 1 \times 4 = 0$$

$$64 + 16c + c^2 - 16 = 0$$

$$c^2 + 16c + 48 = 0$$

$$(c+4)(c+12) = 0$$

$$c = -4 \text{ or } c = -12$$

Taking $c_2 = -4$ and $c_3 = -12$, equations for l_2 and l_3 are $y = 4x - 4$ and $y = 4x - 12$.

c Substituting $c = -4$ into (1) gives

$$t^2 - 4t + 4 = 0$$

$$(t-2)^2 = 0$$

$$t = 2$$

At $t = 2$,

$$x = \frac{t-1}{t} = \frac{2-1}{2} = \frac{1}{2}$$

$$y = t - 4 = 2 - 4 = -2$$

So l_2 meets the curve at $\left(\frac{1}{2}, -2\right)$.

Substituting $c = -12$ into (1) gives

$$t^2 + 4t + 4 = 0$$

$$(t+2)^2 = 0$$

$$t = -2$$

At $t = -2$,

$$x = \frac{-2-1}{-2} = \frac{3}{2}$$

$$y = -2 - 4 = -6$$

So l_3 meets the curve at $\left(\frac{3}{2}, -6\right)$.

Challenge

Find a Cartesian equation for C_1:
$x = e^{2t} \Rightarrow 2t = \ln x$
Substitute into the parametric equation for y:
$y = 2t + 1$
$\therefore\ y = \ln x + 1 \qquad (1)$

Find a Cartesian equations for C_2:
$x = e^{t} \Rightarrow t = \ln x$
Substitute into the parametric equation for y:
$y = 1 + t^2$
$\therefore\ y = 1 + (\ln x)^2 \qquad (2)$

Now solve equations (1) and (2).
Substituting (1) into (2) gives
$\ln x + 1 = 1 + (\ln x)^2$
$(\ln x)^2 - \ln x = 0$
$\ln x(\ln x - 1) = 0$
so $\ln x = 0$ or $\ln x = 1$
$x = e^0 = 1$ or $x = e^1 = e$

Substitute these x-values into either (1) or (2):
When $x = 1,\ y = \ln 1 + 1 = 0 + 1 = 1$
When $x = e,\ y = \ln e + 1 = 1 + 1 = 2$

$\therefore$ the coordinates of the points of intersection are (1, 1) and (e, 2).