Parametric equations 8A

1 a $x = t - 2$

so $t = x + 2$ (1)

$y = t^2 + 1$ (2)

Substitute (1) into (2):

$y = (x+2)^2 + 1$

$= x^2 + 4x + 4 + 1$

$\therefore\ y = x^2 + 4x + 5$

$x = t - 2,\ -4 \le t \le 4$

So the domain of f(x) is $-6 \le x \le 2$.

$y = t^2 + 1,\ -4 \le t \le 4$

So the range of f(x) is $1 \le y \le 17$.

b $x = 5 - t$

so $t = 5 - x$ (1)

$y = t^2 - 1$ (2)

Substitute (1) into (2):

$y = (5-x)^2 - 1$

$= 25 - 10x + x^2 - 1$

$\therefore\ y = x^2 - 10x + 24$

$x = 5 - t,\ t \in \mathbb{R}$

So the domain of f(x) is $x \in \mathbb{R}$.

$y = t^2 - 1,\ t \in \mathbb{R}$

So the range of f(x) is $y \ge -1$.

c $x = \frac{1}{t}$

so $t = \frac{1}{x}$ (1)

$y = 3 - t$ (2)

Substitute (1) into (2):

$y = 3 - \frac{1}{x}$

$x = \frac{1}{t},\ t \ne 0$

So the domain of f(x) is $x \ne 0$.

$y = 3 - t,\ t \ne 0$

Range of f(x) is $y \ne 3$.

d $x = 2t + 1$

so $t = \frac{x-1}{2}$ (1)

$y = \frac{1}{t}$ (2)

Substitute (1) into (2):

$y = \frac{1}{\frac{x-1}{2}}$

$y = \frac{2}{x-1}$

$x = 2t + 1,\ t > 0$

So the domain of f(x) is $x > 1$.

$y = \frac{1}{t},\ t > 0$

So the range of f(x) is $y > 0$.

e $x = \frac{1}{t-2}$

so $t - 2 = \frac{1}{x}$

$t = 2 + \frac{1}{x}$ (1)

$y = t^2$ (2)

Substitute (1) into (2):

$y = \left(2 + \frac{1}{x}\right)^2$

$y = \left(\frac{2x+1}{x}\right)^2$

$x = \frac{1}{t-2},\ t > 2$

So the domain of f(x) is $x > 0$.

$y = t^2,\ t > 2$

So the range of f(x) is $y > 4$.

1 f $x = \dfrac{1}{t+1}$

so $t + 1 = \dfrac{1}{x}$

$t = \dfrac{1}{x} - 1 \qquad (1)$

$y = \dfrac{1}{t-2} \qquad (2)$

Substitute (1) into (2):

$$y = \frac{1}{\frac{1}{x} - 1 - 2}$$

$$= \frac{1}{\frac{1}{x} - 3}$$

$$= \frac{1}{\frac{1-3x}{x}}$$

$$\therefore\ y = \frac{x}{1-3x}$$

$x = \dfrac{1}{t+1}$, $t > 2$

So the domain of f(x) is $0 < x < \dfrac{1}{3}$

$y = \dfrac{1}{t-2}$, $t > 2$, $t > 2$

So the range of f(x) is $y > 0$.

2 a i $x = 2\ln(5-t)$

$\frac{1}{2}x = \ln(5-t)$

$e^{\frac{1}{2}x} = 5 - t$

So $t = 5 - e^{\frac{1}{2}x}$

Substitute $t = 5 - e^{\frac{1}{2}x}$ into $y = t^2 - 5$:

$$y = (5 - e^{\frac{1}{2}x})^2 - 5$$

$$= 25 - 10e^{\frac{1}{2}x} + e^x - 5$$

$$= 20 - 10e^{\frac{1}{2}x} + e^x$$

$x = 2\ln(5-t)$, $t < 4$

When $t = 4$, $x = 2\ln 1 = 0$

and as t increases $2\ln(5-t)$ decreases.

So the range of the parametric function for x is $x > 0$.

Hence the Cartesian equation is

$y = 20 - 10e^{\frac{1}{2}x} + e^x$, $x > 0$

ii $y = t^2 - 5$, $t < 4$

$y = t^2 - 5$ is a quadratic function with minimum value −5 at $t = 0$.

So the range of the parametric function for y is $y \geq -5$.

Hence the range of f(x) is $y \geq -5$.

b i $x = \ln(t+3)$

$e^x = t + 3$

$e^x - 3 = t$

Substitute $t = e^x - 3$ into $y = \dfrac{1}{t+5}$:

$$y = \frac{1}{e^x - 3 + 5} = \frac{1}{e^x + 2}$$

$x = \ln(t+3)$, $t > -2$

When $t = -2$, $x = \ln 1 = 0$

and as t increases, $\ln(t+3)$ increases.

So the range of the parametric function for x is $x > 0$.

Hence the Cartesian equation is

$y = \dfrac{1}{e^x + 2}$, $x > 0$

2 b ii $y = \frac{1}{t+5}$, $t > -2$

When $t = -2$, $y = \frac{1}{3}$

and as t increases, $\frac{1}{t+5}$ decreases towards zero.

So the range of the parametric function for y is $0 < y < \frac{1}{3}$

Hence the range of f(x) is $0 < y < \frac{1}{3}$

c i $x = e^t$

So $y = e^{3t} = (e^t)^3 = x^3$

(Note that since y is a power of x there is no need to substitute for t.)

$x = e^t$, $t \in \mathbb{R}$

The range of the parametric function for x is $x > 0$.

Hence the Cartesian equation is

$y = x^3$, $x > 0$

ii $y = e^{3t}$, $t \in \mathbb{R}$

The range of the parametric function for y is $y > 0$.

Hence the range of f(x) is $y > 0$.

3 a $x = \sqrt{t}$

so $x^2 = t$

Substitute $t = x^2$ into $y = t(9-t)$:

$y = x^2(9 - x^2)$

$= 9x^2 - x^4$

$x = \sqrt{t}$, $0 \le t \le 5$

The range of the parametric function for x is $0 \le x \le \sqrt{5}$.

Hence the Cartesian equation is

$y = 9x^2 - x^4$, $0 \le x \le \sqrt{5}$

$y = t(9-t)$, $0 \le t \le 5$

When $t = 0$, $y = 0$;

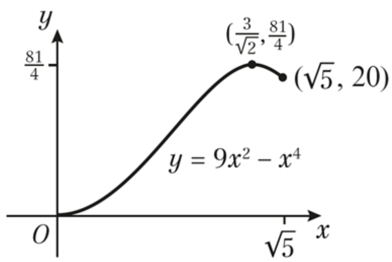
when $t = 5$, $y = 20$;

and $y = t(9-t)$ is a quadratic function with maximum value $\frac{81}{4}$ at $t = \frac{9}{2}$

So the range of the parametric function for y is $0 \le y \le \frac{81}{4}$

Hence the range of f(x) is $0 \le y \le \frac{81}{4}$

b



4 a i $x = 2t^2 - 3$

$x + 3 = 2t^2$

$\frac{x+3}{2} = t^2$

$\pm\sqrt{\frac{x+3}{2}} = t$

Take the positive root since $t > 0$.

Substitute $t = \sqrt{\frac{x+3}{2}}$ into $y = 9 - t^2$:

$y = 9 - \left(\sqrt{\frac{x+3}{2}}\right)^2 = 9 - \frac{x+3}{2}$

$= \frac{18 - x - 3}{2} = \frac{15 - x}{2}$

The Cartesian equation is $y = \frac{15}{2} - \frac{1}{2}x$

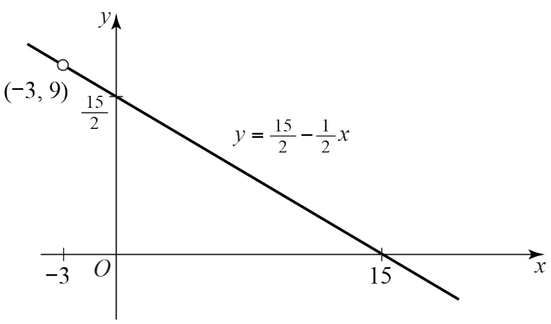
ii $x = 2t^2 - 3,\ t > 0$

$2t^2 - 3$ is a quadratic function with minimum value -3 at $t = 0$.
The range of the parametric function for x is $x > -3$.
Hence the domain of f(x) is $x > -3$.

$y = 9 - t^2,\ t > 0$

$y = 9 - t^2$ is a quadratic function with maximum value 9 at $t = 0$.
So the range of the parametric function for y is $y < 9$.
Hence the range of f(x) is $y < 9$.

iii



b i $x = 3t - 1$

$x + 1 = 3t$

$\frac{x+1}{3} = t$

Substitute $t = \frac{x+1}{3}$ into $y = (t-1)(t+2)$:

$y = \left(\frac{x+1}{3} - 1\right)\left(\frac{x+1}{3} + 2\right)$

$= \left(\frac{x+1-3}{3}\right)\left(\frac{x+1+6}{3}\right)$

$= \left(\frac{x-2}{3}\right)\left(\frac{x+7}{3}\right)$

The Cartesian equation is

$y = \frac{1}{9}(x-2)(x+7)$

ii $x = 3t - 1,\ -4 < t < 4$

When $t = -4$, $x = -13$;
when $t = 4$, $x = 11$.
The range of the parametric function for x is $-13 < x < 11$.
So the domain of f(x) is $-13 < x < 11$.

$y = (t-1)(t+2),\ -4 < t < 4$

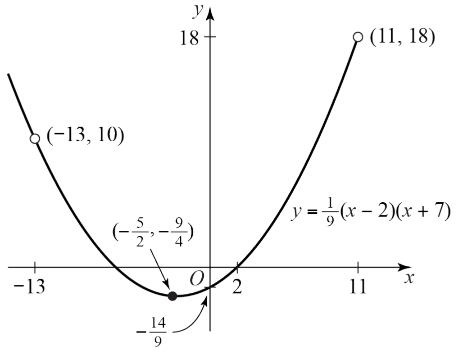
When $t = -4$, $y = 10$;
when $t = 4$, $y = 18$;
and $(t-1)(t+2)$ is a quadratic function with minimum value $-\frac{9}{4}$ at $t = -0.5$.

The range of the parametric function for y is $-\frac{9}{4} \le y < 18$.

Hence the range of f(x) is $-\frac{9}{4} \le y < 18$.

Note: Due to symmetry, the minimum value of y occurs midway between the roots $t = 1$ and $t = -2$, i.e. at $t = -0.5$.

4 b iii



c i $x = t + 1$

$x - 1 = t$

Substitute $t = x - 1$ into $y = \dfrac{1}{t-1}$:

$y = \dfrac{1}{x-1-1} = \dfrac{1}{x-2}$

The Cartesian equation is

$y = \dfrac{1}{x-2}$

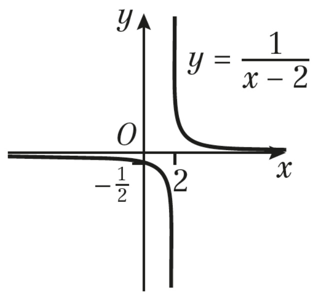
ii $x = t + 1,\ \ t \in \mathbb{R},\ t \neq 1$

So the domain of f(x) is $x \in \mathbb{R},\ x \neq 2$.

$y = \dfrac{1}{t-1},\ t \in \mathbb{R},\ t \neq 1$

So the range of f(x) is $y \in \mathbb{R},\ y \neq 0$.

iii



d i $x = \sqrt{t} - 1$

$x + 1 = \sqrt{t}$

$(x+1)^2 = t$

Substitute $t = (x+1)^2$ into $y = 3\sqrt{t}$:

$y = 3\sqrt{(x+1)^2} = 3(x+1)$

The Cartesian equation is $y = 3x + 3$

ii $x = \sqrt{t} - 1,\ t > 0$

When $t = 0$, $x = -1$

and as t increases $\sqrt{t} - 1$ increases.

The range of the parametric function for x is $x > -1$.

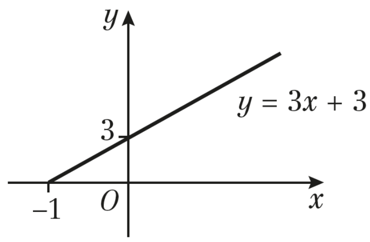
So the domain of f(x) is $x > -1$.

$y = 3\sqrt{t},\ t > 0$

The range of the parametric function for y is $y > 0$.

So the range of f(x) is $y > 0$.

d iii



e i $x = \ln(4 - t)$

$e^x = 4 - t$

$t = 4 - e^x$

Substitute $t = 4 - e^x$ into $y = t - 2$:

$y = 4 - e^x - 2 = 2 - e^x$

The Cartesian equation is $y = 2 - e^x$

ii $x = \ln(4 - t),\ t < 3$

When $t = 3$, $x = \ln 1 = 0$

and as t decreases $\ln(4 - t)$ increases.

So the domain of f(x) is $x > 0$.

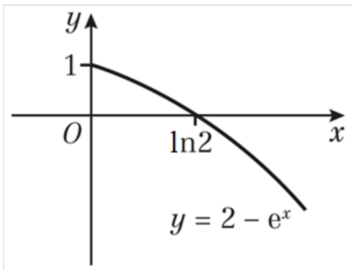
$y = t - 2,\ t < 3$

When $t = 3$, $y = 1$

and as t decreases $t - 2$ decreases.

So the range of f(x) is $y < 1$.

4 e iii



5 a C_1: $x = 1 + 2t$

$\Rightarrow \frac{x-1}{2} = t$

Substitute $t = \frac{x-1}{2}$ into $y = 2 + 3t$:

$y = 2 + 3\left(\frac{x-1}{2}\right)$

$= \frac{4+3x-3}{2} = \frac{3x+1}{2}$

So the Cartesian equation of C_1 is

$y = \frac{3}{2}x + \frac{1}{2}$

C_2: $x = \frac{1}{2t-3}$

$2t - 3 = \frac{1}{x}$

$2t = 3 + \frac{1}{x} = \frac{3x+1}{x}$

$\therefore\ t = \frac{3x+1}{2x}$

and $y = \frac{t}{2t-3} = t\left(\frac{1}{2t-3}\right)$

Substitute $t = \frac{3x+1}{2x}$ and $x = \frac{1}{2t-3}$

into $y = t\left(\frac{1}{2t-3}\right)$:

$y = \left(\frac{3x+1}{2x}\right)x = \frac{3x+1}{2}$

So the Cartesian equation of C_2 is

$y = \frac{3}{2}x + \frac{1}{2}$

Therefore C_1 and C_2 are segments of the same line $y = \frac{3}{2}x + \frac{1}{2}$

5 b For the length of each segment find the domain and range of C_1 and C_2.

For C_1: $x = 1 + 2t,\ 2 < t < 5$

When $t = 2$, $x = 5$;
when $t = 5$, $x = 11$.
The range of the parametric function for x is $5 < x < 11$,
so the domain of C_1 is $5 < x < 11$.

$y = 2 + 3t,\ 2 < t < 5$

When $t = 2$, $y = 8$;
when $t = 5$, $y = 17$.
The range of the parametric function for y is $8 < y < 17$,
so the range of C_1 is $8 < y < 17$.

The endpoints of C_1 have coordinates (5, 8) and (11, 17).

$$\therefore \text{length of } C_1 = \sqrt{(11-5)^2 + (17-8)^2} = \sqrt{36+81} = \sqrt{117} = 3\sqrt{13}$$

For C_2: $x = \dfrac{1}{2t-3},\ 2 < t < 3$

When $t = 2$, $x = 1$;
when $t = 3$, $x = \dfrac{1}{3}$.
The range of the parametric function for x is $\dfrac{1}{3} < x < 1$,
so the domain of C_2 is $\dfrac{1}{3} < x < 1$.

$y = \dfrac{t}{2t-3},\ 2 < t < 3$

When $t = 2$, $y = 2$;
when $t = 3$, $y = 1$.
The range of the parametric function for y is $1 < y < 2$,
so the range of C_2 is $1 < y < 2$.
The endpoints of C_2 have coordinates \`
$\left(\dfrac{1}{3}, 1\right)$ and (1, 2).

$$\therefore \text{length of } C_2 = \sqrt{\left(1-\frac{1}{3}\right)^2 + (2-1)^2} = \sqrt{\frac{4}{9}+1} = \sqrt{\frac{4+9}{9}} = \frac{\sqrt{13}}{3}$$

6 a $x = \dfrac{3}{t} + 2,\ t \neq 0$

The range of the parametric function for x is $x \neq 2$.
(This is also the domain of the Cartesian equation $y = \mathrm{f}(x)$.)

$y = 2t - 3 - t^2,\ t \neq 0$

When $t = 0$, $y = -3$;
$2t - 3 - t^2$ is a quadratic function with maximum value -2 at $t = 1$.
The range of the parametric function for y is $y \leq -2,\ y \neq -3$.
(This is also the range of the Cartesian equation $y = \mathrm{f}(x)$.)

Note: To find the maximum point of the quadratic $y = 2t - 3 - t^2$,

either solve $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$

$$2 - 2t = 0$$
$$2 = 2t$$
$$t = 1$$
$$\therefore y = 2(1) - 3 - (1)^2 = -2$$

or complete the square

$$y = -\left((t-1)^2 - 1 + 3\right) = -\left((t-1)^2 + 2\right) = -(t-1)^2 - 2$$

6 b $x = \frac{3}{t} + 2$

$x - 2 = \frac{3}{t}$

$t = \frac{3}{x-2}$

Substitute $t = \frac{3}{x-2}$ into $y = 2t - 3 - t^2$:

$$y = 2\left(\frac{3}{x-2}\right) - 3 - \left(\frac{3}{x-2}\right)^2$$
$$= \frac{6(x-2) - 3(x-2)^2 - 3^2}{(x-2)^2}$$
$$= -3\left(\frac{-2(x-2) + (x-2)^2 + 3}{(x-2)^2}\right)$$
$$= -3\left(\frac{-2x + 4 + x^2 - 4x + 4 + 3}{(x-2)^2}\right)$$
$$= \frac{-3(x^2 - 6x + 11)}{(x-2)^2}$$

This is a Cartesian equation in the form

$y = \frac{A(x^2 + bx + c)}{(x-2)^2}$ with

$A = -3$, $b = -6$ and $c = 11$.

7 a $x = \ln(t+3)$, $t > -2$

$e^x = t + 3$

$e^x - 3 = t$

Substitute $t = e^x - 3$ into $y = \frac{1}{t+5}$:

$$y = \frac{1}{e^x - 3 + 5} = \frac{1}{e^x + 2}$$

When $t = -2$, $x = \ln 1 = 0$

and as t increases $\ln(t+3)$ increases.

The range of the parametric function for x is $x > 0$,

so the domain of f(x) is $x > 0$.

Therefore the Cartesian equation is

$y = \frac{1}{e^x + 2}$, $x > k$ where $k = 0$.

b $y = \frac{1}{t+5}$, $t > -2$

When $t = -2$, $y = \frac{1}{3}$

and as t increases, $\frac{1}{t+5}$ decreases towards zero.

The range of the parametric function for y is $0 < y < \frac{1}{3}$

so the range of f(x) is $0 < y < \frac{1}{3}$

8 a $x = 3\sqrt{t}$

$\frac{x}{3} = \sqrt{t}$

$\frac{x^2}{9} = t$

Substitute $t = \frac{x^2}{9}$ into $y = t^3 - 2t$:

$$y = \left(\frac{x^2}{9}\right)^3 - 2\left(\frac{x^2}{9}\right) = \frac{x^6}{729} - \frac{2x^2}{9}$$

The Cartesian equation is

$y = \frac{x^6}{729} - \frac{2x^2}{9}$

$x = 3\sqrt{t}$, $0 \le t \le 2$

When $t = 0$, $x = 0$;

when $t = 2$, $x = 3\sqrt{2}$.

The range of the parametric function for x is $0 \le x \le 3\sqrt{2}$

so the domain of f(x) is $0 \le x \le 3\sqrt{2}$.

b $\frac{dy}{dt} = 3t^2 - 2$

$\frac{dy}{dt} = 0$ when $3t^2 - 2 = 0$

$3t^2 = 2$

$t^2 = \frac{2}{3}$

$t = \sqrt{\frac{2}{3}}$ (as $0 \le t \le 2$)

8 c $\frac{d^2y}{dt^2} = 6t$

When $t = \sqrt{\frac{2}{3}}$, $\frac{d^2y}{dt^2} = 6\left(\sqrt{\frac{2}{3}}\right) > 0$

So $t = \sqrt{\frac{2}{3}}$ gives a minimum point of the parametric function for y.
The minimum value of y is

$$\left(\sqrt{\frac{2}{3}}\right)^3 - 2\left(\sqrt{\frac{2}{3}}\right)$$

$$= \frac{2}{3}\sqrt{\frac{2}{3}} - \frac{6}{3}\sqrt{\frac{2}{3}} = -\frac{4}{3}\frac{\sqrt{2}}{\sqrt{3}} = -\frac{4\sqrt{6}}{9}$$

When $t = 0$, $y = 0$;
when $t = 2$, $y = 4$.
The range of the parametric function for y is $-\frac{4\sqrt{6}}{9} \le y \le 4$.
Therefore the range of f(x) is

$$-\frac{4\sqrt{6}}{9} \le \mathrm{f}(x) \le 4.$$

9 a $x = t^3 - t = t(t^2 - 1)$

$\Rightarrow x^2 = t^2(t^2-1)^2$ (1)

$y = 4 - t^2 \Rightarrow t^2 = 4 - y$ (2)

Substitute (2) into (1):

$x^2 = (4-y)(4-y-1)^2$

$x^2 = (4-y)(3-y)^2$

This is in the form $x^2 = (a-y)(b-y)^2$ with $a = 4$ and $b = 3$.

b $y = 4 - t^2$, $t \in \mathbb{R}$

This is a quadratic function of t, and (by symmetry) the maximum value of y occurs at $t = 0$, where $y = 4$.
So 4 is the maximum y-coordinate.

Challenge

a Squaring the parametric functions gives

$x^2 = \left(\frac{1-t^2}{1+t^2}\right)^2$ (1)

$y^2 = \left(\frac{2t}{1+t^2}\right)^2$ (2)

Add (1) and (2):

$$x^2 + y^2 = \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2$$

$$= \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2}$$

$$= \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2}$$

$$= \frac{1+2t^2+t^4}{(1+t^2)^2}$$

$$= \frac{(1+t^2)^2}{(1+t^2)^2} = 1$$

So a Cartesian equation for curve C is $x^2 + y^2 = 1$.

b $x^2 + y^2 = 1$

$\Rightarrow (x-0)^2 + (y-0)^2 = 1$

Curve C is the equation of a circle with centre (0, 0) and radius 1.