Trigonometry and modelling 7G

1 a The maximum height is at 0.25 m when $\sin(1800t)° = 1$

b $0.25\sin(1800t)° = 0.1$

$\sin(1800t)° = \frac{0.1}{0.25} = 0.4$

$1800t = 23.578$ (3 d.p.)

$t = 0.013099$ minutes

$= 0.8$ seconds (1 d.p.)

c The minimum height is at −0.25 m when $\sin(1800t)° = -1$

This occurs when $1800t = 270, 630$
$t = 0.15, 0.35$ minutes
Interval = 0.35 – 0.15 = 0.2 minutes
= 12 seconds

2 a The maximum displacement is at 0.03 radians when $\cos(25t) = 1$

b After 0.2 seconds

$\theta = 0.03\cos(25\times 0.2) = 0.03\times 0.28366$

$= 0.0085$ radians (2 s.f.)

c At $t = 0$, $\theta = 0.03\cos(25\times 0) = 0.03$
To find when $\theta = 0.03$, solve
$0.03\cos(25t) = 0.03$
$\Rightarrow \cos(25t) = 1$
$\Rightarrow 25t = 0, 2\pi, 4\pi, \ldots$
$\Rightarrow t = 0, 0.251, 0.503, \ldots$ (3 d.p.)
The pendulum is first back to its starting position after 0.251 seconds.

d Solve $0.03\cos(25t) = 0.01$

$\cos(25t) = \frac{1}{3}$, $0 \le 25t \le 12.5$

$25t = 1.231, 2\pi - 1.231, 2\pi + 1.231, 4\pi - 1.231$

$25t = 1.231, 5.052, 7.514, 11.335$ (3 d.p.)

$t = 0.0492, 0.2021, 0.3006, 0.4534$ secs

3 a Beginning price when $t = 0$ is
$17.4 + 2\sin(0.7\times 0 - 3) = £17.12$
End price when $t = 9$ is
$17.4 + 2\sin(0.7\times 9 - 3) = £17.08$

b The maximum price of the stock is when $\sin(0.7t - 3) = 1$, so $17.4 + 2 = £19.40$

This is when $\sin(0.7t - 3) = 1$

$0.7t - 3 = \frac{\pi}{2}$

$t = 6.5297$ (4 d.p.)

$t = 6$ hours 32 minutes

c Trader will show a £0.40 profit when
$17.4 + 2\sin(0.7t - 3) = £17.12 + £0.40$
$\Rightarrow \sin(0.7t - 3) = 0.06$

$\Rightarrow 0.7t - 3 = 0.060$ (3 d.p.)

$t = 4.371$

So trader should sell 4 hours 22 minutes after the market opens.

4 a The minimum temperature of the oven is when $\sin(2x - 3) = 1$
$T = 225 - 0.3 = 224.7$ °C

b Solve $225 - 0.3\sin(2x - 3) = 224.7$, for $0 \le x \le 10$

$\Rightarrow \sin(2x - 3) = 1$,

for $-1.5 \le 2x - 3 \le 17$

$2x - 3 = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

$x = 2.285, 5.427, 8.569$ (3 d.p.)

$x = 2\text{ m }17\text{ s}, 5\text{ m }26\text{ s}, 8\text{ m }34\text{ s}$

c $225 - 0.3\sin(2x - 3) = 225.2$, for $x \ge 0$

$\Rightarrow \sin(2x - 3) = -\frac{2}{3}$, for $2x - 3 \ge -3$

$2x - 3 = -2.412, -0.730\ldots$ (3 d.p.)

So oven first reaches minimum temperature at $2x - 3 = -2.412$, so $x = 0.294$ minutes, which is 17.6 seconds (1 d.p.)

5 a Set $0.3\sin\theta - 0.4\cos\theta \equiv R\sin(\theta-\alpha)$
$\equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

So $R\cos\alpha = 0.3$ and $R\sin\alpha = 0.4$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{0.4}{0.3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ \text{ (2 d.p.)}$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 0.3^2 + 0.4^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 0.25$$

$$R = 0.5$$

So $0.3\sin\theta - 0.4\cos\theta = 0.5\sin(\theta - 53.13^\circ)$

b i The maximum value of $0.3\sin\theta - 0.4\cos\theta$ is when $\sin(\theta - 53.13^\circ) = 1$, so the maximum value is 0.5

ii Solve $\sin(\theta - 53.13^\circ) = 1$, for the interval $-53.13^\circ < \theta - 53.13^\circ < 126.87^\circ$

$$\theta - 53.13^\circ = 90^\circ$$

$$\theta = 143.13^\circ$$

c Using part (a)

$23 + 0.3\sin(18x)^\circ - 0.4\cos(18x)^\circ$
$\equiv 23 + 0.5\sin(18x - 53.13^\circ)$

The minimum occurs when $\sin(18x - 53.13^\circ) = -1$

So the minimum temperature is $23 - 0.5 = 22.5\,^\circ\text{C}$

It occurs when $18x - 53.13 = 270$, $x = 17.95$ minutes (2 d.p.)

d At exactly 23 °C, $23 + 0.5\sin(18x - 53.13^\circ) = 23$

Find solutions for $0.5\sin(18x - 53.13)^\circ = 0$ for $0 \le x \le 60$, i.e. in the interval $-53.13 \le 18x - 53.13 \le 1026.87$

So $18x - 53.13 = 0, 180, 360, 540, 720, 900$

Solutions are 3, 13, 23, 33, 43, 53 minutes (nearest minute)

6 a Set $65\cos\theta - 20\sin\theta \equiv R\cos(\theta+\alpha)$
$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

So $R\cos\alpha = 65$ and $R\sin\alpha = 20$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{20}{65}$$

$$\alpha = \tan^{-1}\left(\frac{20}{65}\right) = 0.2985 \text{ (4 d.p.)}$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 65^2 + 20^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 4625$$

$$R = 68.0074 \text{ (4 d.p.)}$$

So
$65\cos\theta - 20\sin\theta$
$= 68.0074\cos(\theta + 0.2985)$

b Using part (a)

$70 - 65\cos 0.2t - 20\sin 0.2t$
$\equiv 70 - 68.0074\cos(0.2t + 0.2985)$

The maximum height is when $\cos(0.2t + 0.2985) = -1$

So $H = 70 + 68.0074 = 138.0\,\text{m}$ (1 d.p.)

c Find consecutive times that the tourist is at the maximum height. This is when

$$\cos(0.2t + 0.2985) = -1$$

$$0.2t + 0.2985 = \pi, 3\pi$$

$$t = 14.216, 45.631 \text{ (3 d.p.)}$$

The time for one revolution is $45.631 - 14.216 = 31.4$ minutes (1 d.p.)

d Find the times the tourist is at 100 m

$$70 - 68.0074\cos(0.2t + 0.2985) = 100$$

$$\cos(0.2t + 0.2985) = -\frac{30}{68.0074} = -0.4411$$

$$0.2t + 0.2985 = 2.0277, 2\pi - 2.0277$$

$$0.2t + 0.2985 = 1.1139, 4.2555$$

$$t = 8.646, 19.785 \text{ (3 d.p.)}$$

Between these times the tourist is above 100 m because the highest point is reached at $t = 14.216$ minutes.

So time spent above 100 m in each revolution = 19.785 – 8.646
= 11.1 minutes (1 d.p.)

7 a Set $200\sin\theta - 150\cos\theta \equiv R\sin(\theta-\alpha)$

$$\equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

So $R\cos\alpha = 200$ and $R\sin\alpha = 150$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{150}{200}$$

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 0.6435 \text{ (4 d.p.)}$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 200^2 + 150^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 62500$$

$$R = 250$$

So $200\sin\theta - 150\cos\theta$

$$\equiv 250\sin(\theta - 0.6435)$$

b i $1700 + 200\sin\left(\frac{4\pi x}{25}\right) - 150\cos\left(\frac{4\pi x}{25}\right)$

$$\equiv 1700 + 250\sin\left(\frac{4\pi x}{25} - 0.6435\right)$$

The maximum value of E is when

$$\sin\left(\frac{4\pi x}{25} - 0.6435\right) = 1$$

So maximum value of E is
$1700 + 250 = 1950$ V/m

ii This maximum occurs when

$$\sin\left(\frac{4\pi x}{25} - 0.6435\right) = 1$$

Look for solutions in the interval

$$-0.6435 \le \frac{4\pi x}{25} - 0.6435 < 4\pi - 0.6435$$

$$\frac{4\pi x}{25} - 0.6435 = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\frac{4\pi x}{25} = 2.2143, 8.4975$$

$$4\pi x = 55.3575, 212.4375$$

$$x = 4.41\text{cm}, 16.91\text{cm (2 d.p.)}$$

c Solve

$$1700 + 250\sin\left(\frac{4\pi x}{25} - 0.6435\right) = 1800$$

for the same interval as in part b ii

$$\sin\left(\frac{4\pi x}{25} - 0.6435\right) = \frac{100}{250} = 0.4$$

$$\frac{4\pi x}{25} - 0.6435 = 0.4115, \pi - 0.4115, 2\pi + 0.4115, 3\pi - 0.4115$$

$$= 0.4115, 2.7301, 6.6947, 9.0133$$

$$x = 2.10, 6.71, 14.60, 19.21 \text{ (2 d.p.)}$$

These results show where E = 1800 V/m. So, because of the shape of the sine curve, $E \le 1800$ V/m when $2.10 \le x \le 6.71$ and $14.60 \le x \le 19.21$

Challenge

a Energy $\propto E^2$ and Energy $\propto \frac{1}{t}$, so $E^2 = \frac{k}{t}$

When E = 1950, t = 20 seconds

$$k = 1950^2 \times 20 = 76\,050\,000$$

When t = 30, E = 1592.1683 V/m

Find where E = 1592.1683 V/m, using the formula for E from question 7:

$$1592.1683 = 1700 + 250\sin\left(\frac{4\pi x}{25} - 0.6435\right)$$

$$-0.4313 = \sin\left(\frac{4\pi x}{25} - 0.6435\right)$$

$$\frac{4\pi x}{25} - 0.6435 = -0.4460, 3.5876, 5.8372, 9.8707, 12.1204$$

This gives these results $0 \le x < 0.393$cm,
$8.42\text{cm} < x < 12.9\text{cm}$ and
$20.9\text{cm} < x < 25\text{cm}$

b Two limitations of the model are
(i) assumes that the field strength is the same from the front to the back of the microwave and (ii) the microwave oven would not necessarily work exactly the same every time it is used.