## Trigonometry and modelling 7E

**1** $5\sin\theta + 12\cos\theta \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

Comparing $\sin\theta$: $R\cos\alpha = 5$

Comparing $\cos\theta$: $R\sin\alpha = 12$

Divide the equations:

$\frac{\sin\alpha}{\cos\alpha} = \frac{12}{5} \Rightarrow \tan\alpha = 2\frac{2}{5}$

Square and add the equations:

$R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2 + 12^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 13^2$

$R = 13$

since $\cos^2\alpha + \sin^2\alpha \equiv 1$

**2** $\sqrt{3}\sin q + \sqrt{6}\cos q$

$\equiv 3\cos\theta\cos\alpha + 3\sin\theta\sin\alpha$

Comparing $\sin\theta: \sqrt{3} = 3\sin\alpha$ (1)

Comparing $\cos\theta: \sqrt{6} = 3\cos\alpha$ (2)

Divide (1) by (2):

$\tan\alpha = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$

So $\alpha = 35.3°$ (1 d.p.)

**3** $2\sin q - \sqrt{5}\cos q$

$\equiv -3\cos\theta\cos\alpha + 3\sin\theta\sin\alpha$

Comparing $\sin\theta: 2 = 3\sin\alpha$ (1)

Comparing $\cos\theta: +\sqrt{5} = +3\cos\alpha$ (2)

Divide (1) by (2):

$\tan\alpha = \frac{2}{\sqrt{5}}$

So $\alpha = 41.8°$ (1 d.p.)

**4 a** Let $\cos\theta - \sqrt{3}\sin\theta \equiv R\cos(\theta + \alpha)$

$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Compare $\cos\theta: R\cos\alpha = 1$ (1)

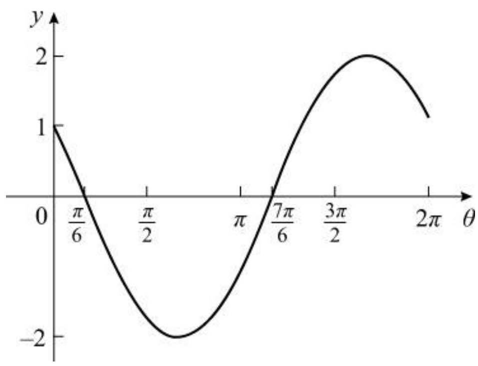
Compare $\sin\theta: R\sin\alpha = \sqrt{3}$ (2)

Divide (2) by (1): $\tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

Square and add: $R^2 = 1 + 3 = 4 \Rightarrow R = 2$

So $\cos\theta - \sqrt{3}\sin\theta \equiv 2\cos\left(\theta + \frac{\pi}{3}\right)$

**b** This is the graph of $y = \cos q$, translated by $\frac{\pi}{3}$ to the left and then stretched in the $y$ direction by scale factor 2.




Meets $y$-axis at (0, 1)

Meets $x$-axis at $\left(\frac{\pi}{6}, 0\right)$, $\left(\frac{7\pi}{6}, 0\right)$

**5 a** Let $7\cos\theta - 24\sin\theta \equiv R\cos(\theta + \alpha)$

$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Compare $\cos\theta$: $R\cos\alpha = 7$ (1)

Compare $\sin\theta$: $R\sin\alpha = 24$ (2)

Divide (2) by (1): $\tan\alpha = \frac{24}{7}$

$\Rightarrow \alpha = 73.7°$ (1 d.p.)

Square and add: $R^2 = 24^2 + 7^2$

$\Rightarrow R = 25$

So $7\cos\theta - 24\sin\theta \equiv 25\cos(\theta + 73.7°)$

**b** Graph meets $y$-axis where $q = 0$,

i.e. $y = 7\cos 0° - 24\sin 0° = 7$

so coordinates are (0, 7)

**c** Maximum value of $25\cos(\theta + 73.7°)$ is when $\cos(\theta + 73.7°) = 1$

So maximum is 25

Minimum value is $25(-1) = -25$

**5 d i** The line $y = 15$ will meet the graph twice in $0 < \theta < 360°$, so there are 2 solutions.

**ii** As the maximum value is 25 it can never be 26, so there are 0 solutions.

**iii** As $-25$ is a minimum, line $y = -25$ only meets curve once, so only 1 solution.

**6 a** Let $\sin\theta + 3\cos\theta \equiv R\sin(\theta+\alpha)$

$$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

Comparing $\sin\theta$: $R\cos\alpha = 1$ (1)

Comparing $\cos\theta$: $R\sin\alpha = 3$ (2)

Divide (2) by (1)

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = 3$$

So $\alpha = 71.56°$ (2 d.p.)

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 3^2$$
$$R^2(\cos^2\alpha + \sin^2\alpha) = 10$$
$$R^2 = 10$$
$$R = \sqrt{10},\ \alpha = 71.6°\ (1\text{ d.p.})$$

**b** Use the value of $\alpha$ to 2 d.p. in calculating values of $\theta$ to avoid rounding errors

$$\sqrt{10}\sin(\theta + 71.56°) = 2$$
$$\sin(\theta + 71.56°) = \frac{2}{\sqrt{10}}$$
$$\sin^{-1}\left(\frac{2}{\sqrt{10}}\right) = 39.23°\ (2\text{ d.p.})$$

As $0 \le \theta < 360°$, the interval for $(\theta + 71.56°)$ is

$$71.56° \le \theta + 71.56° < 431.56°$$

So $\theta + 71.56° = 180° - 39.23°$,

and $\theta + 71.56° = 360° + 39.23°$

$$\theta + 71.56° = 140.77°, 399.23°$$
$$\theta = 69.2°, 327.7°\ (1\text{ d.p.})$$

**7 a** Set $\cos 2\theta - 2\sin 2\theta \equiv R\cos(2\theta + \alpha)$

$$\cos 2\theta - 2\sin 2\theta \equiv R\cos 2\theta\cos\alpha - R\sin 2\theta\sin\alpha$$

Comparing $\sin 2\theta$: $R\sin\alpha = 2$ (1)

Comparing $\cos 2\theta$: $R\cos\alpha = 1$ (2)

Divide (1) by (2)

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = 2$$

So $\alpha = 1.107$ (3 d.p.)

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 2^2$$
$$R^2(\cos^2\alpha + \sin^2\alpha) = 5$$
$$R = \sqrt{5}$$

So $\cos 2\theta - 2\sin 2\theta = \sqrt{5}\cos(2\theta + 1.107)$

**b** $\sqrt{5}\cos(2\theta + 1.107) = -1.5$

$$\cos(2\theta + 1.107) = -\frac{1.5}{\sqrt{5}}$$
$$\cos^{-1}\left(-\frac{1.5}{\sqrt{5}}\right) = 2.306\ (3\text{ d.p.})$$

As $0 \le \theta < \pi$, the interval for $(2\theta + 1.107)$ is

$$1.107 \le 2\theta + 1.107 < 2\pi + 1.107$$

So $2\theta + 1.107 = 2.306,\ 2\pi - 2.306$

$$2\theta + 1.107 = 2.306, 3.977$$
$$\theta = 0.60, 1.44\ (2\text{ d.p.})$$

**8 a** Write $6\sin x + 8\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$, $0 < \alpha < 90°$

So $6\sin x + 8\cos x$
$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

Compare $\sin x$: $R\cos\alpha = 6$ (1)
Compare $\cos x$: $R\sin\alpha = 8$ (2)

Divide (2) by (1): $\tan\alpha = \frac{4}{3}$

$\Rightarrow \alpha = 53.13°$ (2 d.p.)

$R^2 = 6^2 + 8^2 \Rightarrow R = 10$

So $6\sin x + 8\cos x \equiv 10\sin(x + 53.13°)$

Solve $10\sin(x+53.13°) = 5\sqrt{3}$,
in the interval $0 \le x \le 360°$

so $\sin(x+53.13°) = \frac{\sqrt{3}}{2}$

$\Rightarrow x + 53.13° = 60°,\ 120°$

$\Rightarrow x = 6.9°,\ 66.9°$ (1 d.p.)

**b** Let $2\cos 3\theta - 3\sin 3\theta \equiv R\cos(3\theta + \alpha)$
$\equiv R\cos 3\theta\cos\alpha - R\sin 3\theta\sin\alpha$

Compare $\cos 3\theta$: $R\cos\alpha = 2$ (1)
Compare $\sin 3\theta$: $R\sin\alpha = 3$ (2)

Divide (2) by (1): $\tan\alpha = \frac{3}{2}$

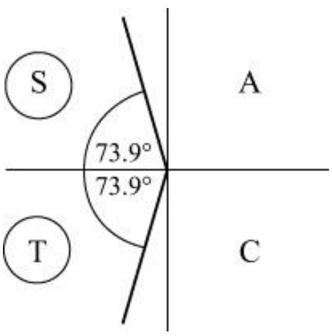
$\Rightarrow \alpha = 56.31°$ (2 d.p.)

$R^2 = 2^2 + 3^2 \Rightarrow R = \sqrt{13}$

Solve $\sqrt{13}\cos(3\theta + 56.31°) = -1$,
in the interval $0 \le \theta \le 90°$

so $\cos(3\theta + 56.31°) = -\frac{1}{\sqrt{13}}$

for $56.31° \le 3\theta + 56.31° \le 326.31°$




$\Rightarrow 3\theta + 56.31° = 106.10°,\ 253.90°$

$\Rightarrow 3\theta = 49.8°,\ 197.6°$

$\Rightarrow \theta = 16.6°,\ 65.9°$ (1 d.p.)

**c** Let $8\cos\theta + 15\sin\theta \equiv R\cos(\theta - \alpha)$
$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

Compare $\cos\theta$: $R\cos\alpha = 8$ (1)
Compare $\sin\theta$: $R\sin\alpha = 15$ (2)

Divide (2) by (1): $\tan\alpha = \frac{15}{8}$

$\Rightarrow \alpha = 61.93°$ (2 d.p.)

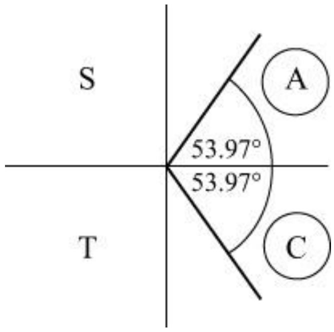
$R^2 = 8^2 + 15^2 \Rightarrow R = 17$

Solve $17\cos(\theta - 61.93°) = 10$,
in the interval $0 \le \theta \le 360°$

So $\cos(\theta - 61.93°) = \frac{10}{17}$,

$-61.93° \le \theta - 61.93° \le 298.07°$

$\cos^{-1}\left(\frac{10}{17}\right) = 53.97°$ (2 d.p.)




So $\theta - 61.93° = -53.97°,\ +53.97°$

$\Rightarrow \theta = 8.0°,\ 115.9°$ (1 d.p.)

**8 d** Let $5\sin\frac{x}{2} - 12\cos\frac{x}{2}$

$$\equiv R\sin\frac{x}{2} - \alpha$$

$$\equiv R\sin\frac{x}{2}\cos\alpha - R\cos\frac{x}{2}\sin\alpha$$

Compare $\sin\frac{x}{2}$: $R\cos\alpha = 5$ (1)

Compare $\cos\frac{x}{2}$: $R\sin\alpha = 12$ (2)

Divide (2) by (1) : $\tan\alpha = \frac{12}{5}$

$\Rightarrow \alpha = 67.38°$ (2 d.p.)

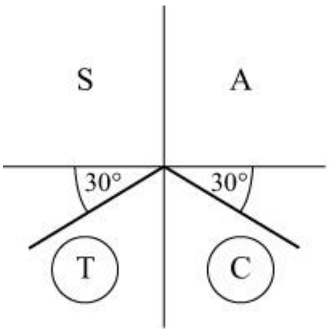
$R = 13$

Solve $13\sin\left(\frac{x}{2} - 67.38°\right) = -6.5$,

in the interval $-360° \le x \le 360°$

So $\sin\left(\frac{x}{2} - 67.38°\right) = -\frac{1}{2}$,

$-247.4° \le \frac{x}{2} - 67.4° \le 112.6°$




From quadrant diagram:

$\frac{x}{2} - 67.38° = -150°, \ -30°$

$\Rightarrow \frac{x}{2} = -82.62°, \ 37.38°$

$\Rightarrow x = -165.2°, \ 74.8°$ (1 d.p.)

**9 a** Set $3\sin 3\theta - 4\cos 3\theta \equiv R\sin(3\theta - \alpha)$

$3\sin 3\theta - 4\cos 3\theta$

$$\equiv R\sin 3\theta\cos\alpha - R\cos 3\theta\sin\alpha$$

Compare $\sin 3\theta$: $R\cos\alpha = 3$ (1)

Compare $\cos 3\theta$: $R\sin\alpha = 4$ (2)

Divide (2) by (1) : $\tan\alpha = \frac{4}{3}$

$\Rightarrow \alpha = 53.13°$ (2 d.p.)

$R^2 = 3^2 + 4^2 = 25 \Rightarrow R = 5$

So $3\sin 3\theta - 4\cos 3\theta \equiv 5\sin(3\theta - 53.13°)$

**b** The minimum value of $3\sin 3\theta - 4\cos 3\theta$ is −5. This occurs when

$\sin(3\theta - 53.13°) = -1$

$3\theta - 53.13° = 270°$

$\theta = 107.7°$ (1 d.p.)

**c** $5\sin(3\theta - 53.13°) = 1$,

in the interval $0 \le \theta < 180°$

So $\sin(3\theta - 53.13°) = \frac{1}{5}$,

in the interval

$-53.13° \le 3\theta - 53.13° < 506.87°$

$3\theta - 53.13° = 11.54°, 168.46°, 371.54°$

$\theta = 21.6°, 73.9°, 141.6°$ (1 d.p.)

**10 a** As $\sin^2 q = \frac{1 - \cos 2q}{2}$ and

$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

So $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$

$$\equiv 5\frac{1 - \cos 2\theta}{2} - 3\frac{1 + \cos 2\theta}{2}$$

$$+ 3(2\sin\theta\cos\theta)$$

$$\equiv \frac{5}{2} - \frac{5}{2}\cos 2\theta - \frac{3}{2} - \frac{3}{2}\cos 2\theta + 3\sin 2\theta$$

$$\equiv 1 - 4\cos 2\theta + 3\sin 2\theta$$

**10 b** Write $3\sin 2q - 4\cos 2q$ in the form $R\sin(2\theta - \alpha)$

The maximum value of $R\sin(2\theta-\alpha)$ is $R$

The minimum value of $R\sin(2\theta-\alpha)$ is $-R$

You know that $R^2 = 3^2 + 4^2$ so $R = 5$

So maximum value of

$1 - 4\cos 2\theta + 3\sin 2\theta$ is $1 + 5 = 6$

and minimum value of

$1 - 4\cos 2\theta + 3\sin 2\theta$ is $1 - 5 = -4$

**c** $1 - 4\cos 2\theta - 3\sin 2\theta = -1$

$\Rightarrow 3\sin 2\theta - 4\cos 2\theta = -2$

Write $3\sin 2q - 4\cos 2q$ in the form $R\sin(2\theta - \alpha)$

So $R\sin(2\theta - \alpha) = -2$

$\Rightarrow 5\sin(2\theta - 53.13°) = -2$

(By solving in same way as Question 9, part a)

Look for solutions in the interval $-53.13° \le 2\theta - 53.13° < 306.87°$

$2\theta - 53.13° = -23.58, 203.58$

$\theta = 14.8°, 128.4°$ (1 d.p.)

**11 a** Let $3\cos q + \sin q ° R\cos(q - a)$

$\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Compare $\cos\theta$: $R\cos\alpha = 3$ (1)

Compare $\sin\theta$: $R\sin\alpha = 1$ (2)

Divide (2) by (1): $\tan\alpha = \frac{1}{3}$

$\Rightarrow \alpha = 18.43°$ (2 d.p.)

$R^2 = 3^2 + 1^2 = 10 \Rightarrow R = \sqrt{10} = 3.16$

Solve $\sqrt{10}\cos(\theta - 18.43°) = 2$,

in the interval $0 \le \theta \le 360°$

$\Rightarrow \cos(\theta - 18.43°) = \frac{2}{\sqrt{10}}$

$\Rightarrow \theta - 18.43° = 50.77°, 309.23°$

$\Rightarrow \theta = 69.2°, 327.7°$ (1 d.p.)

**b** Squaring $3\cos q = 2 - \sin q$

gives $9\cos^2\theta = 4 + \sin^2\theta - 4\sin\theta$

$\Rightarrow 9(1 - \sin^2\theta) = 4 + \sin^2\theta - 4\sin\theta$

$\Rightarrow 10\sin^2\theta - 4\sin\theta - 5 = 0$

**c** $10\sin^2 q - 4\sin q - 5 = 0$

$\Rightarrow \sin\theta = \frac{4 \pm \sqrt{216}}{20}$

For $\sin\theta = \frac{4 + \sqrt{216}}{20}$, $\sin\theta$ is positive,

so $\theta$ is in the first and second quadrants.

$\Rightarrow \theta = 69.2°, 180° - 69.2°$

$= 69.2°, 110.8°$ (1 d.p.)

For $\sin\theta = \frac{4 - \sqrt{216}}{20}$, $\sin\theta$ is negative,

so $\theta$ is in the third and fourth quadrants.

$\Rightarrow \theta = 180° - (-32.3°), 360° + (-32.3°)$

$= 212.3°, 327.7°$ (1 d.p.)

So solutions of quadratic in (b) are

$69.2°, 110.8°, 212.3°, 327.7°$ (1 d.p.)

**d** In squaring the equation, you are also including the solutions to

$3\cos q = -(2 - \sin q)$,

which when squared produces the same quadratic. The extra two solutions satisfy this equation.

**12 a** $\cot\theta + 2 = \operatorname{cosec}\theta$

$$\Rightarrow \frac{\cos\theta}{\sin\theta} + 2 = \frac{1}{\sin\theta}$$

Multiplying both sides by $\sin\theta$ gives

$\cos\theta + 2\sin\theta = 1$

**b** $\cos\theta + 2\sin\theta = 1$

Set $2\sin\theta + \cos\theta \equiv R\sin(\theta+\alpha)$
$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

So $R\cos\alpha = 1$ and $R\sin\alpha = 1$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{1}{2}$$

$\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 26.57°$ (2 d.p.)

$R^2 = 2^2 + 1^2 \Rightarrow R = \sqrt{5}$

So $\sqrt{5}\sin(\theta + 26.57)° = 1$

$\sin(\theta + 26.57)° = \frac{1}{\sqrt{5}}$, in the interval

$26.57° \le \theta + 26.57° < 386.57°$

$\theta + 26.57 = 26.57, 153.43$

$\theta = 0°, 126.9°$ (1 d.p.)

As both $\cot\theta$ and $\operatorname{cosec}\theta$ are undefined at 0, $\theta = 126.9°$ is the only solution.

**13 a** $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + \left(\sqrt{3} - 1\right)\sin\theta = 2$

$$\Rightarrow \sqrt{2}\cos\theta\cos\frac{\pi}{4} + \sqrt{2}\sin\theta\sin\frac{\pi}{4} + \sqrt{3}\sin\theta - \sin\theta = 2$$

$$\Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) + \sqrt{3}\sin\theta - \sin\theta = 2$$

$\Rightarrow \cos\theta + \sin\theta + \sqrt{3}\sin\theta - \sin\theta = 2$

$\Rightarrow \cos\theta + \sqrt{3}\sin\theta = 2$

**b** $\cos\theta + \sqrt{3}\sin\theta = 2$

Set $\sqrt{3}\sin\theta + \cos\theta \equiv R\sin(\theta+\alpha)$
$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

So $R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$R^2 = \sqrt{3}^2 + 1^2 = 4 \Rightarrow R = 2$

$$2\sin\left(\theta + \frac{\pi}{6}\right) = 2$$

$\sin\left(\theta + \frac{\pi}{6}\right) = 1$, in the interval

$$\frac{\pi}{6} \le \theta + \frac{\pi}{6} \le \frac{11\pi}{6}$$

$$\theta + \frac{\pi}{6} = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{3}$$

**14 a** Set $9\cos\theta + 40\sin\theta \equiv R\cos(\theta - \alpha)$
$\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

So $R\cos\alpha = 9$ and $R\sin\alpha = 40$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{40}{9}$$

$$\alpha = \tan^{-1}\left(\frac{40}{9}\right)$$

So $\alpha = 77.320°$ (3 d.p.)

$R^2\cos^2\alpha + R^2\sin^2\alpha = 40^2 + 9^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 1681$

$R = 41$

So $9\cos\theta + 40\sin\theta = 41\cos(\theta - 77.320°)$

**b i** $$g(\theta) = \frac{18}{50 + 41\cos(\theta - 77.320°)}$$

The minimum value of $g(\theta)$ is when $\cos(\theta - 77.320°) = 1$

So the minimum value is $\frac{18}{50 + 41} = \frac{18}{91}$

**14 b ii** The minimum occurs when

$\cos(\theta - 77.320°) = 1$

$\theta - 77.320° = 0$

$\theta = 77.320°$

**15 a** Set $12\cos 2\theta - 5\sin 2\theta \equiv R\cos(2\theta + \alpha)$

$\equiv R\cos 2\theta \cos\alpha - R\sin 2\theta \sin\alpha$

So $R\cos\alpha = 12$ and $R\sin\alpha = 5$

$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{5}{12}$

$\alpha = \tan^{-1}\left(\frac{5}{12}\right)$

So $\alpha = 22.62°$ (2 d.p.)

$R^2\cos^2\alpha + R^2\sin^2\alpha = 12^2 + 5^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 169$

$R = 13$

**b** $13\cos(2\theta + 22.62°) = -6.5$

$\cos(2\theta + 22.62°) = -\frac{6.5}{13}$, in the interval

$22.62° \leq 2\theta + 22.62° < 382.62°$

$2\theta + 22.62° = 120°, 240°$

$\theta = 48.7°, 108.7°$ (1 d.p.)

**c** $24\cos^2\theta - 10\sin\theta\cos\theta$

$\equiv 24\left(\frac{\cos 2\theta + 1}{2}\right) - 5\sin 2\theta$

$\equiv 12\cos 2\theta - 5\sin 2\theta + 12$

a = 12, b = −5 and c = 12

**d** $24\cos^2\theta - 10\sin\theta\cos\theta$

$\equiv 12\cos 2\theta - 5\sin 2\theta + 12$

From part (a)

$12\cos 2\theta - 5\sin 2\theta + 12$

$= 13\cos(2\theta + 22.62°) + 12$

The minimum value is therefore when

$\cos(2\theta + 22.620°) = -1$

It is $13(-1) + 12 = -1$