Trigonometry and modelling 7B

1 a $\cos 15° = \cos(45° - 30°)$
$= \cos 45° \cos 30° + \sin 45° \sin 30°$
$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$
$= \frac{\sqrt{2}\sqrt{3} + \sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$

b $\sin 75° = \sin(45° + 30°)$
$= \sin 45° \cos 30° + \cos 45° \sin 30°$
$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$
$= \frac{\sqrt{2}\sqrt{3} + \sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$

Note $\sin 75° = \cos(90° - 75°) = \cos 15°$

c $\sin(120° + 45°)$
$= \sin 120° \cos 45° + \cos 120° \sin 45°$
$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right) \times \frac{\sqrt{2}}{2}$
$= \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$

d $\tan 165° = \tan(120° + 45°)$
$= \frac{\tan 120° + \tan 45°}{1 - \tan 120° \tan 45°}$
$\tan 120° = \frac{\sin 120°}{\cos 120°} = \frac{\sin 60°}{-\cos 60°}$
$= -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$

So $\tan 120° = \frac{-\sqrt{3}+1}{1+\sqrt{3}}$
$= \frac{(1-\sqrt{3}+1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$
$= \frac{-4+2\sqrt{3}}{2}$
$= -2 + \sqrt{3}$

2 a Using $\sin(A+B)$ expansion
$\sin 30° \cos 60° + \cos 30° \sin 60°$
$= \sin(30° + 60°) = \sin 90° = 1$

b $\cos 110° \cos 20° + \sin 110° \sin 20°$
$= \cos(110° - 20°) = \cos 90° = 0$

c $\sin 33° \cos 27° + \cos 33° \sin 27°$
$= \sin(33° + 27°) = \sin 60° = \frac{\sqrt{3}}{2}$

d $\cos\frac{\pi}{8}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\sin\frac{\pi}{8}$
$= \cos\left(\frac{\pi}{8} + \frac{\pi}{8}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

e $\sin 60° \cos 15° - \cos 60° \sin 15°$
$= \sin(60° - 15°) = \sin 45° = \frac{\sqrt{2}}{2}$

f $\cos 70° \cos 50° - \cos 70° \tan 70° \sin 50°$
$= \cos 70° \cos 50° - \sin 70° \sin 50°$

Simplifying as
$\left(\cos\theta \times \tan\theta = \cancel{\cos\theta} \times \frac{\sin\theta}{\cancel{\cos\theta}} = \sin\theta\right)$

So $\cos 70°(\cos 50° - \tan°\sin 50°)$
$= \cos(70° + 50°)$
$= \cos 120° = -\cos 60° = -\frac{1}{2}$

g $\frac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°}$
$= \tan(45+15)° = \tan 60° = \sqrt{3}$

h Use the fact that $\tan 45° = 1$ to rewrite as
$\frac{1 - \tan 15°}{1 + \tan 15°} = \frac{\tan 45° - \tan 15°}{1 + \tan 45° \tan 15°}$
$= \tan(45° - 15°) = \tan 30° = \frac{\sqrt{3}}{3}$

i $\frac{\tan\frac{7\pi}{12} - \tan\frac{\pi}{3}}{1 + \tan\frac{7\pi}{12}\tan\frac{\pi}{3}} = \tan\left(\frac{7\pi}{12} - \frac{\pi}{3}\right)$
$= \tan\frac{3\pi}{12} = \tan\frac{\pi}{4} = 1$

2 j This is very similar to part (e) but to appreciate this you need to rewrite the equation as

$$\sqrt{3}\cos 15° - \sin 15°$$
$$\equiv 2\left(\frac{\sqrt{3}}{2}\cos 15° - \frac{1}{2}\sin 15°\right)$$
$$\equiv 2(\sin 60°\cos 15° - \cos 60°\sin 15°)$$
$$\equiv 2\sin(60-15)°$$
$$\equiv 2\sin 45°$$
$$= \sqrt{2}$$

3 a $\tan(45°+30°) = \dfrac{\tan 45° + \tan 30°}{1-\tan 45°\tan 30°}$

b $\tan 75° = \dfrac{1+\frac{\sqrt{3}}{3}}{1-\frac{\sqrt{3}}{3}}$

$$= \frac{3+\sqrt{3}}{3-\sqrt{3}} = \frac{\left(3+\sqrt{3}\right)\left(3+\sqrt{3}\right)}{\left(3-\sqrt{3}\right)\left(3+\sqrt{3}\right)}$$
$$= \frac{12+6\sqrt{3}}{6} = 2+\sqrt{3}$$

4 $\cot(A+B) = 2$

$\Rightarrow \tan(A+B) = \dfrac{1}{2}$

$\Rightarrow \dfrac{\tan A + \tan B}{1-\tan A\tan B} = \dfrac{1}{2}$

But as $\cot A = \dfrac{1}{4}$, then $\tan A = 4$.

So $\dfrac{4+\tan B}{1-4\tan B} = \dfrac{1}{2}$

$\Rightarrow 8 + 2\tan B = 1 - 4\tan B$

$\Rightarrow 6\tan B = -7$

$\Rightarrow \tan B = -\dfrac{7}{6}$

So $\cot B = \dfrac{1}{\tan B} = -\dfrac{6}{7}$

5 a $\cos 105° = \cos(45°+60°)$

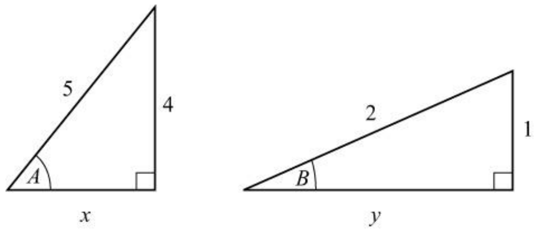
$$= \cos 45°\cos 60° - \sin 45°\sin 60°$$
$$= \frac{1}{\sqrt{2}}\times\frac{1}{2} - \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}$$
$$= \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

b $\sec 105° = \dfrac{1}{\cos 105°}$

$$= \frac{1}{\frac{\sqrt{2}-\sqrt{6}}{4}} = \frac{4}{\sqrt{2}-\sqrt{6}}$$
$$= \frac{4}{\sqrt{2}-\sqrt{6}}\times\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}+\sqrt{6}}$$
$$= \frac{4\left(\sqrt{2}+\sqrt{6}\right)}{-4} = -\sqrt{2}\left(1+\sqrt{3}\right)$$

So a = 2 and b = 3

6 Draw the right-angled triangles containing A and B



Using Pythagoras' theorem gives

$x = 3$ and $y = \sqrt{3}$

a $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \frac{4}{5}\times\frac{\sqrt{3}}{2} + \frac{3}{5}\times\frac{1}{2} = \frac{4\sqrt{3}+3}{10}$$

b $\cos(A - B) = \cos A\cos B + \sin A\sin B$

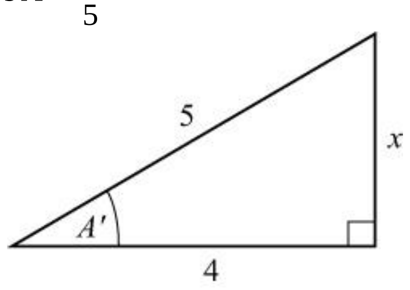
$$= \frac{3}{5}\times\frac{\sqrt{3}}{2} + \frac{4}{5}\times\frac{1}{2} = \frac{3\sqrt{3}+4}{10}$$

c $\sec(A - B) = \dfrac{1}{\cos(A - B)} = \dfrac{10}{3\sqrt{3}+4}$

$$= \frac{10(3\sqrt{3}-4)}{(3\sqrt{3}+4)(3\sqrt{3}-4)}$$
$$= \frac{10(3\sqrt{3}-4)}{27-16}$$
$$= \frac{10(3\sqrt{3}-4)}{11}$$

7 Let $A' = 180° - A$. As A is the second quadrant $\cos A' = -\cos A$

Draw a right-angled triangle where $\cos A' = \frac{4}{5}$



Using Pythagoras' theorem $x = 3$

So $\sin A' = \frac{3}{5}$, $\tan A' = \frac{3}{4}$

a As A is in the second quadrant,

$\sin A = \sin A'$, $\sin A = \frac{3}{5}$

b $\cos(\pi + A) = \cos\pi\cos A - \sin\pi\sin A$

$= -\cos A$

As $\cos\pi = -1$, $\sin\pi = 0$

So $\cos(\pi + A) = \frac{4}{5}$

c $\sin\left(\frac{\pi}{3} + A\right) = \sin\frac{\pi}{3}\cos A + \cos\frac{\pi}{3}\sin A$

$= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{4}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)$

$= \frac{3 - 4\sqrt{3}}{10}$

d As A is in the second quadrant,

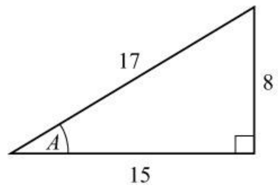
$\tan A = -\tan A' = -\frac{3}{4}$

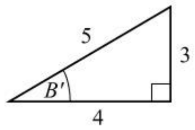
$\tan\left(\frac{\pi}{4} + A\right) = \frac{\tan\frac{\pi}{4} + \tan A}{1 - \tan\frac{\pi}{4}\tan A}$

$= \frac{1 + \tan A}{1 - \tan A} = \frac{\frac{1}{4}}{\frac{7}{4}} = \frac{1}{7}$

8 Let $B' = 180° - B$. As B is in the second quadrant $\cos B' = -\cos B$, $\sin B' = \sin B$ and $\tan B' = -\tan B$.

Drawing right-angled triangles for A and B', use Pythagoras' theorem to find the missing sides, which are 15 and 3.





So $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$

and $\sin B = \frac{3}{5}$, $\cos B = -\frac{4}{5}$, $\tan B = -\frac{3}{4}$

a $\sin(A - B) = \sin A\cos B - \cos A\sin B$

$= \left(\frac{8}{17}\right)\left(-\frac{4}{5}\right) - \left(\frac{15}{17}\right)\left(\frac{3}{5}\right)$

$= \frac{-32 - 45}{85} = -\frac{77}{85}$

b $\cos(A - B) = \cos A\cos B + \sin A\sin B$

$= \left(\frac{15}{17}\right)\left(-\frac{4}{5}\right) + \left(\frac{8}{17}\right)\left(\frac{3}{5}\right)$

$= \frac{-60 + 24}{85} = -\frac{36}{85}$

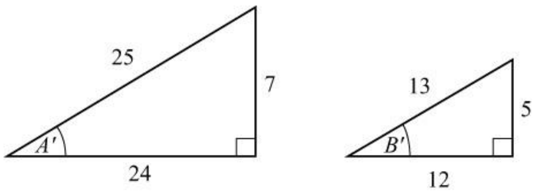
c $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

$= \frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{24}{60}} = \frac{\frac{77}{60}}{\frac{36}{60}} = \frac{77}{36}$

So $\cot(A - B) = \frac{1}{\tan(A - B)} = \frac{36}{77}$

9 Angle A is in the third quadrant as it is reflex and tan A is positive. Let $A' = A - 180°$, so $\sin A = -\sin A'$, $\cos A = -\cos A'$, $\tan A = \tan A'$
Let $B' = 180° - B$. As B is in the second quadrant $\cos B' = -\cos B$, $\sin B' = \sin B$ and $\tan B' = -\tan B$.

Drawing right-angled triangles for A' and B' use Pythagoras' theorem to find the missing sides, which are 25 and 12.



So $\sin A = -\frac{7}{25}$, $\cos A = -\frac{24}{25}$, $\tan A = \frac{7}{24}$
and $\sin B = \frac{5}{13}$, $\cos B = -\frac{12}{13}$, $\tan B = -\frac{5}{12}$

a $\sin(A+B) = \sin A \cos B + \cos A \sin B$
$$= \left(-\frac{7}{25}\right)\left(-\frac{12}{13}\right) + \left(-\frac{24}{25}\right)\left(\frac{5}{13}\right)$$
$$= \frac{84-120}{325} = -\frac{36}{325}$$

b $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
$$= \frac{\frac{7}{24}+\frac{5}{12}}{1-\left(\frac{7}{24}\right)\left(\frac{5}{12}\right)} = \frac{\frac{17}{24}}{\frac{253}{288}} = \frac{204}{253}$$

c $\operatorname{cosec}(A+B) = \frac{1}{\sin(A+B)} = -\frac{325}{36}$

10 a $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
$$= \frac{\frac{1}{5}+\frac{2}{3}}{1-\frac{1}{5}\times\frac{2}{3}} = \frac{\frac{13}{15}}{\frac{15-2}{15}} = \frac{\frac{13}{15}}{\frac{13}{15}} = 1$$

As tan ($A + B$) is positive, $A + B$ is in the first or third quadrants, but as A and B are both acute $A + B$ cannot be in the third quadrant, so $A + B = \tan^{-1} 1 = 45°$

b A is reflex but $\tan A°$ is positive, so A is in the third quadrant, i.e. $180° < A < 270°$ and $0° < B < 90°$. As tan ($A + B$) is positive, $A + B$ is in the first or third quadrants.

As $180° < A + B < 360°$, it must be in the third quadrant, so $A + B = \tan^{-1} 1 = 225°$