Binomial expansion 4C

1 a Let $\frac{8x+4}{(1-x)(2+x)} \equiv \frac{A}{(1-x)} + \frac{B}{(2+x)}$

$$\equiv \frac{A(2+x)+B(1-x)}{(1-x)(2+x)}$$

Set the numerators equal:

$8x + 4 \equiv A(2 + x) + B(1 - x)$

Substitute $x = 1$:

$8 \times 1 + 4 = A \times 3 + B \times 0$
$\Rightarrow 12 = 3A$
$\Rightarrow A = 4$

Substitute $x = -2$:

$8 \times (-2) + 4 = A \times 0 + B \times 3$
$\Rightarrow -12 = 3B$
$\Rightarrow B = -4$

Hence $\frac{8x+4}{(1-x)(2+x)} \equiv \frac{4}{(1-x)} - \frac{4}{(2+x)}$

1 b $\dfrac{4}{(1-x)} = 4(1-x)^{-1}$

$$= 4\left(1+(-1)(-x)+\frac{(-1)(-2)(-x)^2}{2!}+\ldots\right)$$

$$= 4\left(1+x+x^2+\ldots\right)$$

$$= 4+4x+4x^2+\ldots$$

$$\frac{4}{(2+x)} = 4(2+x)^{-1}$$

$$= 4\left(2\left(1+\frac{x}{2}\right)\right)^{-1}$$

$$= 4\times 2^{-1}\left(1+\frac{x}{2}\right)^{-1}$$

$$= 4\times\frac{1}{2}\times\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\ldots\right)$$

$$= 2\left(1-\frac{x}{2}+\frac{x^2}{4}+\ldots\right)$$

$$= 2-x+\frac{1}{2}x^2+\ldots$$

Therefore

$$\frac{8x+4}{(1-x)(2+x)} \equiv \frac{4}{(1-x)}-\frac{4}{(2+x)}$$

$$= \left(4+4x+4x^2+\ldots\right)-\left(2-x+\frac{1}{2}x^2+\ldots\right)$$

$$= 2+5x+\frac{7x^2}{2}+\ldots$$

c $\dfrac{4}{(1-x)}$ is valid for $|x| < 1$

$\dfrac{4}{(2+x)}$ is valid for $|x| < 2$

Both are valid when $|x| < 1$.

2 a Let $\frac{-2x}{(2+x)^2} \equiv \frac{A}{(2+x)} + \frac{B}{(2+x)^2}$

$$\equiv \frac{A(2+x)+B}{(2+x)^2}$$

Set the numerators equal:

$-2x \equiv A(2 + x) + B$

Substitute $x = -2$:

$4 = A \times 0 + B \Rightarrow B = 4$

Equate terms in x:

$-2 = A \Rightarrow A = -2$

Hence $\frac{-2x}{(2+x)} \equiv \frac{-2}{(2+x)} + \frac{4}{(2+x)^2}$

2 b

$$\frac{-2}{2+x} = -2(2+x)^{-1}$$
$$= -2\left(2\left(1+\frac{x}{2}\right)\right)^{-1}$$
$$= -2\times 2^{-1}\times\left(1+\frac{x}{2}\right)^{-1}$$
$$= -1\times\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)$$
$$= -1\times\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$$
$$= -1+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{8}+\ldots$$

$$\frac{4}{(2+x)^2} = 4(2+x)^{-2}$$
$$= 4\left(2\left(1+\frac{x}{2}\right)\right)^{-2}$$
$$= 4\times 2^{-2}\times\left(1+\frac{x}{2}\right)^{-2}$$
$$= 1\times\left(1+(-2)\left(\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)$$
$$= 1\times\left(1-x+\frac{3x^2}{4}-\frac{x^3}{2}+\ldots\right)$$
$$= 1-x+\frac{3x^2}{4}-\frac{x^3}{2}+\ldots$$

Hence

$$\frac{-2x}{(2+x)^2} \equiv \frac{-2}{(2+x)}+\frac{4}{(2+x)^2}$$
$$= -1+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{8}+\ldots+1-x+\frac{3x^2}{4}-\frac{x^3}{2}+\ldots$$
$$= 0-\frac{1}{2}x+\frac{1}{2}x^2-\frac{3}{8}x^3+\ldots$$

Hence $B=\frac{1}{2}$ (coefficient of x^2) and $C=-\frac{3}{8}$ (coefficient of x^3).

2 c $\frac{-2}{(2+x)}$ is valid for $|x| < 2$

$\frac{4}{(2+x)^2}$ is valid for $|x| < 2$

Hence whole expression is valid $|x| < 2$.

3 a Let $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{A}{(1+x)} + \frac{B}{(1-x)} + \frac{C}{(2+x)}$

$$\equiv \frac{A(1-x)(2+x)+B(1+x)(2+x)+C(1+x)(1-x)}{(1+x)(1-x)(2+x)}$$

Set the numerators equal

$6 + 7x + 5x^2 \equiv A(1 - x)(2 + x) + B(1 + x)(2 + x) + C(1 + x)(1 - x)$

Substitute $x = 1$:

$6 + 7 + 5 = A \times 0 + B \times 2 \times 3 + C \times 0$
$\Rightarrow 18 = 6B$
$\Rightarrow B = 3$

Substitute $x = -1$:

$6 - 7 + 5 = A \times 2 \times 1 + B \times 0 + C \times 0$
$\Rightarrow 4 = 2A$
$\Rightarrow A = 2$

Substitute $x = -2$:

$6 - 14 + 20 = A \times 0 + B \times 0 + C \times (-1) \times 3$
$\Rightarrow 12 = -3C$
$\Rightarrow C = -4$

Hence $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{2}{(1+x)} + \frac{3}{(1-x)} - \frac{4}{(2+x)}$

3 b $$\frac{2}{1+x} = 2(1+x)^{-1}$$
$$= 2\left(1+(-1)(x)+\frac{(-1)(-2)(x)^2}{2!}+\frac{(-1)(-2)(-3)(x)^3}{3!}+\ldots\right)$$
$$= 2(1-x+x^2-x^3+\ldots)$$
$$\approx 2-2x+2x^2-2x^3 \quad \text{Valid for } |x|<1$$

$$\frac{3}{1-x} = 3(1-x)^{-1}$$
$$= 3\left(1+(-1)(-x)+\frac{(-1)(-2)(-x)^2}{2!}+\frac{(-1)(-2)(-3)(-x)^3}{3!}+\ldots\right)$$
$$= 3(1+x+x^2+x^3+\ldots)$$
$$\approx 3+3x+3x^2+3x^3 \quad \text{Valid for } |x|<1$$

$$\frac{4}{2+x} = 4(2+x)^{-1}$$
$$= 4\left(2\left(1+\frac{x}{2}\right)\right)^{-1}$$
$$= 4\times 2^{-1}\times\left(1+\frac{x}{2}\right)^{-1}$$
$$= 2\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)$$
$$= 2\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$$
$$\approx 2-x+\frac{x^2}{2}-\frac{x^3}{4} \quad \text{Valid for } |x|<2$$

Hence $$\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{2}{(1+x)}+\frac{3}{(1-x)}-\frac{4}{(2+x)}$$
$$= (2-2x+2x^2-2x^3)+(3+3x+3x^2+3x^3)-\left(2-x+\frac{x^2}{2}-\frac{x^3}{4}\right)+\ldots$$
$$= 2+3-2-2x+3x+x+2x^2+3x^2-\frac{x^2}{2}-2x^3+3x^3+\frac{x^3}{4}+\ldots$$
$$= 3+2x+\frac{9}{2}x^2+\frac{5}{4}x^3+\ldots$$

c All expansions are valid when $|x| < 1$.

4 a $\dfrac{12x-1}{(1+2x)(1-3x)} \equiv \dfrac{A}{1+2x} + \dfrac{B}{1-3x} \equiv \dfrac{A(1-3x)+B(1+2x)}{(1+2x)(1-3x)}$

So $12x-1 \equiv A(1-3x)+B(1+2x)$

Let $x = -\dfrac{1}{2}$:

$-6-1 = A \times \dfrac{5}{2} + 0$

$-7 = \dfrac{5}{2}A$

$A = -\dfrac{14}{5}$

Let $x = \dfrac{1}{3}$:

$4-1 = 0 + B \times \dfrac{5}{3}$

$3 = \dfrac{5}{3}B$

$B = \dfrac{9}{5}$

$A = -\dfrac{14}{5}$, $B = \dfrac{9}{5}$

b $\dfrac{12x-1}{(1+2x)(1-3x)} \equiv \dfrac{-14}{5(1+2x)} + \dfrac{9}{5(1-3x)}$

$\dfrac{-14}{5(1+2x)} = -\dfrac{14}{5}(1+2x)^{-1}$

$= -\dfrac{14}{5}\left(1+(-1)(2x)+\dfrac{(-1)(-2)}{2!}(2x)^2+\ldots\right)$

$= -\dfrac{14}{5}\left(1-2x+4x^2+\ldots\right)$

$= -\dfrac{14}{5}+\dfrac{28}{5}x-\dfrac{56}{5}x^2+\ldots$

$\dfrac{9}{5(1-3x)} = \dfrac{9}{5}(1-3x)^{-1}$

$= \dfrac{9}{5}\left(1+(-1)(-3x)+\dfrac{(-1)(-2)}{2!}(-3x)^2+\ldots\right)$

$= \dfrac{9}{5}\left(1+3x+9x^2+\ldots\right)$

$= \dfrac{9}{5}+\dfrac{27}{5}x+\dfrac{81}{5}x^2+\ldots$

$\dfrac{-14}{5(1+2x)}+\dfrac{9}{5(1-3x)} = -\dfrac{14}{5}+\dfrac{28}{5}x-\dfrac{56}{5}x^2+\dfrac{9}{5}+\dfrac{27}{5}x+\dfrac{81}{5}x^2+\ldots$

$= -1+11x+5x^2+\ldots$

5 a $\frac{2x^2+7x-6}{(x+5)(x-4)} \equiv A + \frac{B}{x+5} + \frac{C}{x-4}$

$$\begin{array}{r} 2 \\ x^2+x-20\overline{)\,2x^2+7x-6} \\ \underline{2x^2+2x-40} \\ 5x+34 \end{array}$$

$A = 2$

$\frac{2x^2+7x-6}{(x+5)(x-4)} \equiv 2 + \frac{5x+34}{(x+5)(x-4)}$

$\frac{5x+34}{(x+5)(x-4)} \equiv \frac{B}{x+5} + \frac{C}{x-4} = \frac{B(x-4)+C(x+5)}{(x+5)(x-4)}$

$5x+34 = B(x-4)+C(x+5)$

Let $x=-5$:

$-25+34 = B\times(-9)+0$

$9 = -9B$

$B = -1$

Let $x = 4$:

$20+34 = 0 + C\times 9$

$54 = 9C$

$C = 6$

$\frac{2x^2+7x-6}{(x+5)(x-4)} \equiv 2 - \frac{1}{x+5} + \frac{6}{x-4}$

b $2 - \frac{1}{x+5} + \frac{6}{x-4} = 2-(5+x)^{-1} + 6(-4+x)^{-1} = 2 - \frac{1}{5}\left(1+\frac{1}{5}x\right)^{-1} - \frac{3}{2}\left(1-\frac{1}{4}x\right)^{-1}$

$\frac{1}{5}\left(1+\frac{1}{5}x\right)^{-1} = \frac{1}{5}\left(1+(-1)\left(\frac{1}{5}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{5}x\right)^2+\ldots\right)$

$= \frac{1}{5}\left(1-\frac{1}{5}x+\frac{1}{25}x^2+\ldots\right)$

$= \frac{1}{5}-\frac{1}{25}x+\frac{1}{125}x^2+\ldots$

$\frac{3}{2}\left(1-\frac{1}{4}x\right)^{-1} = \frac{3}{2}\left(1+(-1)\left(-\frac{1}{4}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{1}{4}x\right)^2+\ldots\right)$

$= \frac{3}{2}\left(1+\frac{1}{4}x+\frac{1}{16}x^2+\ldots\right)$

$= \frac{3}{2}+\frac{3}{8}x+\frac{3}{32}x^2+\ldots$

$2 - \frac{1}{x+5} + \frac{6}{x-4} = 2 - \left(\frac{1}{5}-\frac{1}{25}x+\frac{1}{125}x^2+\ldots\right) - \left(\frac{3}{2}+\frac{3}{8}x+\frac{3}{32}x^2+\ldots\right)$

$= \frac{3}{10} - \frac{67}{200}x + \frac{407}{4000}x^2 + \ldots$

5 c $|x| < 4$

6 a
$$\begin{array}{r} 3 \\ x^2+x-6\overline{)3x^2+4x-5} \\ \underline{3x^2+3x-18} \\ x+13 \end{array}$$

$A = 3$

$$\frac{3x^2+4x-5}{(x+3)(x-2)} \equiv 3 + \frac{x+13}{(x+3)(x-2)}$$

$$\frac{x+13}{(x+3)(x-2)} \equiv \frac{B}{x+3} + \frac{C}{x-2} = \frac{B(x-2)+C(x+3)}{(x+3)(x-2)}$$

$x+13 = B(x-2)+C(x+3)$

Let $x = -3$

$-3+13 = B\times(-5)+0$

$10 = -5B$

$B = -2$

Let $x = 2$:

$2+13 = 0 + C\times 5$:

$15 = 5C$

$C = 3$

$A = 3$, $B = -2$ and $C = 3$

6 b $\frac{3x^2+4x-5}{(x+3)(x-2)} \equiv 3-\frac{2}{x+3}+\frac{3}{x-2}$

$$3-\frac{2}{x+3}+\frac{3}{x-2}=3-2(3+x)^{-1}+3(-2+x)^{-1}=3-\frac{2}{3}\left(1+\frac{1}{3}x\right)^{-1}-\frac{3}{2}\left(1-\frac{1}{2}x\right)^{-1}$$

$$\frac{2}{3}\left(1+\frac{1}{3}x\right)^{-1}=\frac{2}{3}\left(1+(-1)\left(\frac{1}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{3}x\right)^2+\dots\right)$$

$$=\frac{2}{3}\left(1-\frac{1}{3}x+\frac{1}{9}x^2+\dots\right)$$

$$=\frac{2}{3}-\frac{2}{9}x+\frac{2}{27}x^2+\dots$$

$$\frac{3}{2}\left(1-\frac{1}{2}x\right)^{-1}=\frac{3}{2}\left(1+(-1)\left(-\frac{1}{2}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{1}{2}x\right)^2+\dots\right)$$

$$=\frac{3}{2}\left(1+\frac{1}{2}x+\frac{1}{4}x^2+\dots\right)$$

$$=\frac{3}{2}+\frac{3}{4}x+\frac{3}{8}x^2+\dots$$

$$3-\frac{2}{x+3}+\frac{3}{x-2}=3-\left(\frac{2}{3}-\frac{2}{9}x+\frac{2}{27}x^2+\dots\right)-\left(\frac{3}{2}+\frac{3}{4}x+\frac{3}{8}x^2+\dots\right)$$

$$=\frac{5}{6}-\frac{19}{36}x-\frac{97}{216}x^2+\dots$$

7 a $\dfrac{2x^2+5x+11}{(2x-1)^2(x+1)} \equiv \dfrac{A}{2x-1}+\dfrac{B}{(2x-1)^2}+\dfrac{C}{x+1}$

$\equiv \dfrac{A(2x-1)(x+1)+B(x+1)+C(2x-1)^2}{(2x-1)^2(x+1)}$

$2x^2+5x+11 \equiv A(2x-1)(x+1)+B(x+1)+C(2x-1)^2$

Let $x=\frac{1}{2}$:

$\frac{1}{2}+\frac{5}{2}+11=0+B\times\frac{3}{2}+0$

$14=\frac{3}{2}B$

$B=\frac{28}{3}$

Let $x=-1$:

$2-5+11=0+0+C\times 9$

$8=9C$

$C=\frac{8}{9}$

Equating coefficients of x^2 gives:

$2=2A+4C$

$2=2A+\frac{32}{9}$

$A=-\frac{7}{9}$

$A=-\frac{7}{9}$, $B=\frac{28}{3}$ and $C=\frac{8}{9}$

7 b $$\frac{2x^2+5x+11}{(2x-1)^2(x+1)} \equiv \frac{-7}{9(2x-1)} + \frac{28}{3(2x-1)^2} + \frac{8}{9(x+1)}$$

$$-\frac{7}{9}(-1+2x)^{-1} + \frac{28}{3}(-1+2x)^{-2} + \frac{8}{9}(1+x)^{-1}$$

$$= \frac{7}{9}(1-2x)^{-1} + \frac{28}{3}(1-2x)^{-2} + \frac{8}{9}(1+x)^{-1}$$

$$\frac{7}{9}(1-2x)^{-1} = \frac{7}{9}\left(1+(-1)(-2x)+\frac{(-1)(-2)}{2!}(-2x)^2+\ldots\right)$$

$$= \frac{7}{9}\left(1+2x+4x^2+\ldots\right)$$

$$= \frac{7}{9}+\frac{14}{9}x+\frac{28}{9}x^2+\ldots$$

$$\frac{28}{3}(1-2x)^{-2} = \frac{28}{3}\left(1+(-2)(-2x)+\frac{(-2)(-3)}{2!}(-2x)^2+\ldots\right)$$

$$= \frac{28}{3}\left(1+4x+12x^2+\ldots\right)$$

$$= \frac{28}{3}+\frac{112}{3}x+112x^2+\ldots$$

$$\frac{8}{9}(1+x)^{-1} = \frac{8}{9}\left(1+(-1)x+\frac{(-1)(-2)}{2!}x^2+\ldots\right)$$

$$= \frac{8}{9}\left(1-x+x^2+\ldots\right)$$

$$= \frac{8}{9}-\frac{8}{9}x+\frac{8}{9}x^2+\ldots$$

$$\frac{2x^2+5x+11}{(2x-1)^2(x+1)} = \frac{7}{9}+\frac{14}{9}x+\frac{28}{9}x^2+\ldots+\frac{28}{3}+\frac{112}{3}x+112x^2+\ldots+\frac{8}{9}-\frac{8}{9}x+\frac{8}{9}x^2+\ldots$$

$$= 11+38x+116x^2+\ldots$$

c $$\mathrm{f}(0.05) = \frac{2(0.05)^2+5(0.05)+11}{(2(0.05)-1)^2(0.05+1)} = 13.23339212$$

Using the expansion:

$\mathrm{f}(0.05) \approx 11+38(0.05)+116(0.05)^2 = 13.19$

Percentage error = $\dfrac{13.23339212-13.19}{13.23339212}\times 100 = 0.33\%$