Sequences and series 3I

1 a Initial amount = £4000 (start of month 1)

Start of month 2 = £(4000 + 200)

Start of month 3 = £(4000 + 200 + 200)
= £(4000 + 2 × 200)

Start of month 10 = £(4000 + 9 ×200)
= £(4000 + 1800)
= £5800

b Start of mth month
= £(4000 + (m − 1) × 200)
= £(4000 + 200m − 200)
= £(3800 + 200m)

2

$$\underset{\text{Year 1}}{20\,000} \xrightarrow[\text{increment}]{\text{1st}} + \underset{\text{Year 2}}{20\,500} \xrightarrow[\text{increment}]{\text{2nd}} + \underset{\text{Year 3}}{21\,000} \xrightarrow[\text{increment}]{\text{3rd}} + \underset{\text{Year 4}}{21\,500} + \ldots$$

Carol will reach her maximum salary after

$$\frac{25\,000-20\,000}{500}=10 \text{ increments}$$

This will be after 11 years.

a Total amount after 10 years

$$=\underbrace{20\,000 + 20\,500 + 21\,000 + \ldots}$$

This is an arithmetic series with a = 20 000, d = 500 and n = 10. Use $S_n=\frac{n}{2}(2a+(n-1)d)$.

$$=\frac{10}{2}(40\,000+9\times500)$$
$$=5\times44\,500$$
$$=£222\,500$$

b From year 11 to year 15 she will continue to earn £25 000.

Total in this time = 5 × 25 000
= £125 000.

Total amount in the first 15 years is

£222 500 + £125 000 = £347 500

c It is unlikely her salary will rise by the same amount each year.

3 Amount saved by James

$$=\underbrace{1 + 2 + 3 + \ldots + 42}$$

This is an arithmetic series with a = 1, d = 1, n = 42 and L = 42.

a Use $S_n=\frac{n}{2}(a+L)$

$$=\frac{42}{2}(1+42)$$
$$=21\times43$$
$$=903\text{p}$$
$$=£9.03$$

b To save £100 we need

$$\underbrace{1+2+3+\ldots}_{\text{Sum to } n \text{ terms}}=10\,000$$

$$\frac{n}{2}(2\times1+(n-1)\times1)=10\,000$$
$$\frac{n}{2}(n+1)=10\,000$$
$$n(n+1)=20\,000$$
$$n^2+n-20\,000=0$$

$$n=\frac{-1\pm\sqrt{(1)^2-4\times1\times(-20\,000)}}{2}$$
$$n=140.9 \text{ or } -141.9$$

It takes James 141 days to save £100.

4 A growth of 10% a year gives a multiplication factor of 1.1.

a After 1 year number is $200 \times 1.1 = 220$

b After 2 years number is $200 \times 1.1^2 = 242$

c After 3 years number is

$200 \times 1.1^3 = 266.2 = 266$
(to nearest whole number)

d After 10 years number is

$200 \times 1.1^{10} = 518.748\ldots = 519$
(to nearest whole number)

5 Let maximum speed in bottom gear be a km h^{-1}

This gives maximum speeds in each successive gear **of** ar, ar^2, ar^3, where r is the common ratio.

We are given

$a = 40$ **(1)**

$ar^3 = 120$ **(2)**

Substitute (**1**) into (**2**):

$$40r^3 = 120 \quad (\div 40)$$
$$r^3 = 3$$
$$r = \sqrt[3]{3}$$
$$r = 1.442\ldots \ (3 \text{ d.p.})$$

Maximum speed in 2nd gear is
$ar = 40 \times 1.442\ldots = 57.7$ km h^{-1}

Maximum speed in 3rd gear is
$ar^2 = 40 \times (1.442\ldots)^2 = 83.2$ km h^{-1}

6 **a** $r = 0.85$
$a \times 0.85^3 = 11\,054.25$
$a = £18\,000$

b $18\,000 \times 0.85^n > 5000$

$$0.85^n > \frac{5}{18}$$
$$n > \frac{\log\left(\frac{5}{18}\right)}{\log(0.85)}$$
$$n > 7.88$$

7 **a** Total commission

$$= \underbrace{10 + 20 + 30 + \ldots + 520}$$

Arithmetic series with $a = 10$, $d = 10$, $n = 52$.

$$= \frac{52}{2}\left(2\times10+(52-1)\times10\right) \text{ using}$$
$$S_n = \frac{n}{2}\left(2a+(n-1)d\right)$$
$$= 26(20+51\times10)$$
$$= 26(20+510)$$
$$= 26\times530$$
$$= £13\,780$$

b Commission = policies for year 1 + policies for 2nd week of year 2
= 520 + 22 = £542

c Total commission for year 2

= Commission for year 1 policies + Commission for year 2 policies

= 520 × 52 + (11 + 22 + 33 + ... 52 × 11)

Use $S_n = \frac{n}{2} = \left(2a+(n-1)d\right)$
with $n = 52,\ a = 11,\ d = 11$

$$= 27\,040 + \frac{52}{2}\left(2\times11+(52-1)\times11\right)$$
$$= 27\,040 + 26\times(22+51\times11)$$
$$= 27\,040 + £\,15158$$
$$= £\,42198$$

8 a Cost of drilling to 500 m

$$= \underset{\substack{\uparrow \\ \text{1st} \\ 50\,\text{m}}}{500} + \underset{\substack{\uparrow \\ \text{2nd} \\ 50\,\text{m}}}{640} + \underset{\substack{\uparrow \\ \text{3rd} \\ 50\,\text{m}}}{780} + \ldots$$

There would be 10 terms because there are 10 lots of 50 m in 500 m.

Arithmetic series with $a = 500$, $d = 140$ and $n = 10$.

$$\text{Using } S_n = \frac{n}{2}(2a+(n-1)d)$$
$$= \frac{10}{2}(2\times 500+(10-1)\times 140)$$
$$= 5(1000+9\times 140)$$
$$= 5\times 2260$$
$$= £11300$$

b This time we are given $S = 76\,000$. The first term will still be 500 and d remains 140.

Use $S = \frac{n}{2}(2a+(n-1)d)$ with $S = 76\,000$, $a = 500$, $d = 140$, and solve for n.

$$76000 = \frac{n}{2}(2\times 500+(n-1)\times 140)$$
$$76000 = \frac{n}{2}(1000+140(n-1))$$
$$76000 = n(500+70(n-1))$$
$$76000 = n(500+70n-70)$$
$$76000 = n(70n+430n) \quad (\text{multiply out})$$
$$76000 = 70n^2+430n \quad (\div 10)$$
$$7600 = 7n^2+43n$$

$$0 = 7n^2+43n-7600$$
$$n = \frac{-43\pm\sqrt{(43)^2-4\times 7\times(-7600)}}{2\times 7}$$
$$n = 30.02, \ (-36.16)$$

Only accept the positive answer, so there are 30 terms (to the nearest term).

So the greatest depth that can be drilled is 30 × 50 = 1500 m (to the nearest 50 m).

9 a 1st year = 500

2nd year = 550 = 500 + 1 × 50

3rd year = 600 = 500 + 2 × 50

⋮

40th year = 500 + 39 × 50 = £2450

b Total amount paid in

$$= \underbrace{£500 + £550 + £600 + \ldots + £2450}$$

This is an arithmetic series with $a = 500$, $d = 50$, $L = 2450$ and $n = 40$.

$$S_n = \frac{n}{2}(a+L)$$
$$S_{40} = \frac{40}{2}(500+2450)$$
$$= 20\times 2950$$
$$= £59000$$

c Brian's amount

$$= \underbrace{890 + (890+d) + (890+2d) + \ldots}_{40\text{ years}}$$

Use $S_n = \frac{n}{2}(2a+(n-1)d)$ with $n = 40$, $a = 890$ and d.

$$S_{40} = \frac{40}{2}(2\times 890+(40-1)d)$$
$$= 20(1780+39d)$$

Use the fact that

Brian's saving = Anne's savings

$$20(1780+39d) = 59000 \quad (\div 20)$$
$$1780+39d = 2950 \quad (-1780)$$
$$39d = 1170 \quad (\div 39)$$
$$d = 30$$

10 If the number of people infected increases by 4% the multiplication factor is 1.04.

After n days $100 \times (1.04)^n$ people will be infected.

If 1000 people are infected

$$100\times(1.04)^n = 1000$$
$$(1.04)^n = 10$$
$$\log(1.04)^n = \log 10$$
$$n\log(1.04) = 1$$
$$n = \frac{1}{\log(1.04)}$$
$$n = 58.708\ldots$$

It would take 59 days.

11 If the increase is 3.5% per annum the multiplication factor is 1.035.

Therefore after n years I will have £$A \times (1.035)^n$.

If the money is doubled it will equal $2A$, therefore

$$A\times(1.035)^n = 2A$$
$$(1.035)^n = 2$$
$$\log(1.035)^n = \log 2$$
$$n\log(1.035) = \log 2$$
$$n = \frac{\log 2}{\log(1.035)} = 20.14879\ldots$$

My money will double after 20.15 years.

12 The reduction is 6% which gives a multiplication factor of 0.94.

Let the number of fish now be F.

After n years there will be $F \times (0.94)^n$.

When their number is halved the number will be $\frac{1}{2}F$.

Set these equal to each other:

$$F\times(0.94)^n = \frac{1}{2}F$$
$$(0.94)^n = \frac{1}{2}$$
$$\log(0.94)^n = \log\left(\frac{1}{2}\right)$$
$$n\log(0.94) = \log\left(\frac{1}{2}\right)$$
$$n = \frac{\log\left(\frac{1}{2}\right)}{\log(0.94)}$$
$$n = 11.2$$

The fish stocks will halve in 11.2 years.

13 No. grains $= \underbrace{1 + 2 + 4 + 8 + \ldots}_{\text{64 terms}}$

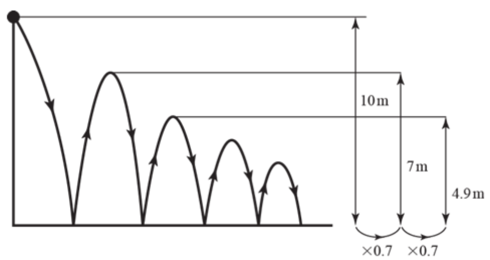
This is a geometric series with $a = 1$, $r = 2$ and $n = 64$.

As $|r| > 1$ use $S_n = \frac{a(r^n - 1)}{r - 1}$.

$$\text{Number of grains} = \frac{1(2^{64}-1)}{2-1} = 2^{64} - 1$$

$$= 1.84\times10^{19}$$

14 a



After the 1st bounce it bounces to 7 cm

After the 2nd bounce it bounces to 4.9 cm (× 0.7)

After the 3rd bounce it bounces to 3.43 cm (× 0.7)

After the 4th bounce it bounces to 2.401 cm (× 0.7)

14 b Total distance travelled

$$= \underbrace{10}_{\substack{\text{1st}\\\text{bounce}}} + 7 + \underbrace{7}_{\substack{\text{2nd}\\\text{bounce}}} + 4.9 + \underbrace{4.9}_{\substack{\text{3rd}\\\text{bounce}}} + \ldots$$

$$= 2\times\underbrace{(10 + 7 + 4.9 + \ldots)}_{\substack{\text{6 terms}\\ a=10,\ r=0.7,\ n=6}} - 10$$

$$= 2\times\frac{10\left(1-0.7^6\right)}{1-0.7} - 10$$

$$= 48.8234\,\text{m}$$

15 a $a = 10$, $r = 1.1$

$$S_n = \frac{10(1.1^n - 1)}{1.1-1} = 1000$$

$1.1^n - 1 = 10$

$1.1^n = 11$

$$n = \frac{\log 11}{\log 1.1}$$

$= 25.16$

So 26 days

b On the 25th day:
$ar^{24} = 10 \times 1.1^{24} = 98.5$ miles

16 Jan. 1st, year 1 = £500

Dec. 31st, year 1 = 500×1.035

Jan. 1st, year 2 = $500 \times 1.035 + 500$

Dec. 31st, year 2
$= (500 \times 1.035 + 500) \times 1.035$
$= 500 \times 1.035^2 + 500 \times 1.035$

⋮

Dec. 31st, year n

$= 500 \times 1.035^n + \ldots + 500 \times 1.035^2$
$+ 500 \times 1.035$

$$= 500\times\underbrace{\left(1.035^n + \ldots + 1.035^2 + 1.035\right)}$$

A geometric series with $a = 1.035$, $r = 1.035$ and n.

Use $S_n = \dfrac{a\left(r^n - 1\right)}{r-1}$.

Dec. 31st year $n = 500\times\dfrac{1.035\left(1.035^n - 1\right)}{1.035-1}$

Set this equal to £20 000.

$$20\,000 = 500\times\frac{1.035\left(1.035^n - 1\right)}{1.035-1}$$

$$\left(1.035^n - 1\right) = \frac{20\,000\times(1.035-1)}{500\times 1.035}$$

$$1.035^n - 1 = 1.3526570\ldots$$

$$1.035^n = 2.3526570\ldots$$

$$\log\left(1.035^n\right) = \log 2.3526570\ldots$$

$$n\log(1.035) = \log 2.3526570\ldots$$

$$n = \frac{\log 2.3526570\ldots}{\log 1.035}$$

$$n = 24.9 \text{ years (3 s.f.)}$$

It takes Alan 25 years to save £20 000.