Sequences and series 3H

1 a i The sequence is increasing.

b i The sequence is decreasing.

c i The sequence is increasing.

d i The sequence is periodic.

ii Order 2

2 a i $u_n = 20 - 3n$
$u_1 = 20 - 3(1) = 17$
$u_2 = 20 - 3(2) = 14$
$u_3 = 20 - 3(3) = 11$
$u_4 = 20 - 3(4) = 8$
$u_5 = 20 - 3(5) = 5$

ii The sequence is decreasing.

b i $u_n = 2^{n-1}$
$u_1 = 2^{1-1} = 1$
$u_2 = 2^{2-1} = 2$
$u_3 = 2^{3-1} = 4$
$u_4 = 2^{4-1} = 8$
$u_5 = 2^{5-1} = 16$

ii The sequence is increasing.

c i $u_n = \cos(180n°)$
$u_1 = \cos(180(1)°) = -1$
$u_2 = \cos(180(2)°) = 1$
$u_3 = \cos(180(3)°) = -1$
$u_4 = \cos(180(4)°) = 1$
$u_5 = \cos(180(5)°) = -1$

ii The sequence is periodic.

iii Order 2

d i $u_n = (-1)^n$
$u_1 = (-1)^1 = -1$
$u_2 = (-1)^2 = 1$
$u_3 = (-1)^3 = -1$
$u_4 = (-1)^4 = 1$
$u_5 = (-1)^5 = -1$

ii The sequence is periodic.

iii Order 2

e i $u_{n+1} = u_n - 5$
$u_1 = 20$
$u_2 = 20 - 5 = 15$
$u_3 = 15 - 5 = 10$
$u_4 = 10 - 5 = 5$
$u_5 = 5 - 5 = 0$

ii The sequence is decreasing.

f i $u_{n+1} = 5 - u_n$
$u_1 = 20$
$u_2 = 5 - 20 = -15$
$u_3 = 5 + 15 = 20$
$u_4 = 5 - 20 = -15$
$u_5 = 5 - 5 = 20$

ii The sequence is periodic.

iii Order 2

g i $u_{n+1} = \frac{2}{3}u_n$
$u_1 = k$
$u_2 = \frac{2k}{3}$
$u_3 = \frac{2}{3}\left(\frac{2k}{3}\right) = \frac{4k}{9}$
$u_4 = \frac{2}{3}\left(\frac{4k}{9}\right) = \frac{8k}{27}$
$u_5 = \frac{2}{3}\left(\frac{8k}{27}\right) = \frac{16k}{81}$

2 g ii The sequence is dependent on the value of k.

3 $u_{n+1} = ku_n$

$u_1 = 5$

$u_2 = 5k$

$u_3 = 5k^2$

If $k \geqslant 1$ the sequence is increasing.

If $k \leqslant 0$ the sequence is periodic.

If $0 < k < 1$ the sequence is decreasing.

4 $u_{n+1} = pu_n + 10$

$u_1 = 5$

$u_2 = 5p + 10$

$u_3 = p(5p + 10) + 10$

As the sequence is periodic with order 2,

$p(5p + 10) + 10 = 5$

$5p^2 + 10p + 5 = 0$

$p^2 + 2p + 1 = 0$

$(p + 1)^2 = 0$

$p = -1$

5 a $a_n = \cos(90n°)$

$a_1 = \cos(90(1)°) = 0$

$a_2 = \cos(90(2)°) = -1$

$a_3 = \cos(90(3)°) = 0$

$a_4 = \cos(90(4)°) = 1$

$a_5 = \cos(90(5)°) = 0$

$a_6 = \cos(90(6)°) = -1$

Order 4

b $\sum_{r=1}^{444} a_r = 111(0 - 1 + 0 + 1) = 0$

Challenge

a $u_{n+2} = \dfrac{1 + u_{n+1}}{u_n}$

$u_1 = a$

$u_2 = b$

$u_3 = \dfrac{1+b}{a}$

$$u_4 = \frac{1 + \frac{1+b}{a}}{b} = \frac{a+b+1}{ab}$$

$$u_5 = \frac{1 + \frac{a+b+1}{ab}}{\frac{1+b}{a}} = \frac{ab+a+b+1}{b(1+b)}$$

$$= \frac{a(b+1)+b+1}{b(1+b)} = \frac{a+1}{b}$$

$$u_6 = \frac{1 + \frac{a+1}{b}}{\frac{a+b+1}{ab}} = \frac{a+b+1}{b} \times \frac{ab}{a+b+1} = a$$

$$u_7 = \frac{1+a}{\frac{a+1}{b}} = (1+a) \times \frac{b}{a+1} = b$$

Therefore, the sequence is periodic and order 5

b When $a = 2$ and $b = 9$

$u_1 = 2$

$u_2 = 9$

$u_3 = \dfrac{1+9}{2} = 5$

$u_4 = \dfrac{2+9+1}{2 \times 9} = \dfrac{2}{3}$

$u_5 = \dfrac{2+1}{9} = \dfrac{1}{3}$

$$\sum_{r=1}^{5} u_r = 2 + 9 + 5 + \frac{2}{3} + \frac{1}{3} = 17$$

Series is periodic so $\sum_{r=1}^{5} u_r = \sum_{r=6}^{10} u_r = \sum_{r=11}^{15} u_r$ and so on.

c So $\sum_{r=1}^{100} u_r = 20 \times \sum_{r=1}^{5} u_r = 20 \times 17 = 340$