### Sequences and series 3H

1. **a i** The sequence is increasing.

2. **a i** \( u_n = 20 - 3n \)
   
   \[
   \begin{align*}
   u_1 &= 20 - 3(1) = 17 \\
   u_2 &= 20 - 3(2) = 14 \\
   u_3 &= 20 - 3(3) = 11 \\
   u_4 &= 20 - 3(4) = 8 \\
   u_5 &= 20 - 3(5) = 5
   \end{align*}
   \]
   
   ii The sequence is decreasing.

3. **b i** The sequence is decreasing.

4. **b i** \( u_n = 2^{n-1} \)
   
   \[
   \begin{align*}
   u_1 &= 2^{1-1} = 1 \\
   u_2 &= 2^{2-1} = 2 \\
   u_3 &= 2^{3-1} = 4 \\
   u_4 &= 2^{4-1} = 8 \\
   u_5 &= 2^{5-1} = 16
   \end{align*}
   \]
   
   ii The sequence is increasing.

5. **c i** The sequence is increasing.

6. **c i** \( u_n = \cos(180n^\circ) \)
   
   \[
   \begin{align*}
   u_1 &= \cos(180(1)^\circ) = -1 \\
   u_2 &= \cos(180(2)^\circ) = 1 \\
   u_3 &= \cos(180(3)^\circ) = -1 \\
   u_4 &= \cos(180(4)^\circ) = 1 \\
   u_5 &= \cos(180(5)^\circ) = -1
   \end{align*}
   \]
   
   ii The sequence is periodic.

7. **d i** \( u_n = (-1)^n \)
   
   \[
   \begin{align*}
   u_1 &= (-1)^1 = -1 \\
   u_2 &= (-1)^2 = 1 \\
   u_3 &= (-1)^3 = -1 \\
   u_4 &= (-1)^4 = 1 \\
   u_5 &= (-1)^5 = -1
   \end{align*}
   \]
   
   ii The sequence is periodic.

8. **e i** \( u_{n+1} = u_n - 5 \)
   
   \[
   \begin{align*}
   u_1 &= 20 \\
   u_2 &= 20 - 5 = 15 \\
   u_3 &= 15 - 5 = 10 \\
   u_4 &= 10 - 5 = 5 \\
   u_5 &= 5 - 5 = 0
   \end{align*}
   \]
   
   ii The sequence is decreasing.

9. **f i** \( u_{n+1} = 5 - u_n \)
   
   \[
   \begin{align*}
   u_1 &= 20 \\
   u_2 &= 5 - 20 = -15 \\
   u_3 &= 5 + 15 = 20 \\
   u_4 &= 5 - 20 = -15 \\
   u_5 &= 5 - 5 = 0
   \end{align*}
   \]
   
   ii The sequence is periodic.

10. **g i** \( u_{n+1} = \frac{2}{3} u_n \)
   
   \[
   \begin{align*}
   u_1 &= k \\
   u_2 &= \frac{2k}{3} \\
   u_3 &= \frac{2}{3} \left( \frac{2k}{3} \right) = \frac{4k}{9} \\
   u_4 &= \frac{2}{3} \left( \frac{4k}{9} \right) = \frac{8k}{27} \\
   u_5 &= \frac{2}{3} \left( \frac{8k}{27} \right) = \frac{16k}{81}
   \end{align*}
   \]
   
   ii The sequence is periodic.

iii Order 2
The sequence is dependent on the value of \( k \).

\[ u_{n+1} = ku_n \]

\[ u_1 = 5 \]
\[ u_2 = 5k \]
\[ u_3 = 5k^2 \]

If \( k \geq 1 \) the sequence is increasing.
If \( k \leq 0 \) the sequence is periodic.
If \( 0 < k < 1 \) the sequence is decreasing.

\[ u_{n+1} = pu_n + 10 \]

\[ u_1 = 5 \]
\[ u_2 = 5p + 10 \]
\[ u_3 = p(5p + 10) + 10 \]

As the sequence is periodic with order 2,
\[ p(5p + 10) + 10 = 5 \]
\[ 5p^2 + 10p + 5 = 0 \]
\[ p^2 + 2p + 1 = 0 \]
\[ (p + 1)^2 = 0 \]
\[ p = -1 \]

\[ a_n = \cos(90n^\circ) \]
\[ a_1 = \cos(90(1)^\circ) = 0 \]
\[ a_2 = \cos(90(2)^\circ) = -1 \]
\[ a_3 = \cos(90(3)^\circ) = 0 \]
\[ a_4 = \cos(90(4)^\circ) = 1 \]
\[ a_5 = \cos(90(5)^\circ) = 0 \]
\[ a_6 = \cos(90(6)^\circ) = -1 \]

Order 4

\[ \sum_{r=1}^{444} a_r = 111(0 - 1 + 0 + 1) = 0 \]

\[ a \quad u_{n+2} = \frac{1 + u_{n+1}}{u_n} \]
\[ u_1 = a \]
\[ u_2 = b \]
\[ u_3 = \frac{1 + b}{a} \]
\[ u_4 = \frac{1 + \frac{1 + b}{a}}{b} = \frac{a + b + 1}{ab} \]
\[ u_5 = \frac{1 + \frac{a + b + 1}{b}}{a + b + 1} = \frac{ab + a + b + 1}{b(a + b + 1)} \]
\[ u_6 = \frac{1 + \frac{ab + a + b + 1}{b(a + b + 1)}}{a + b + 1} = \frac{a}{b} \]
\[ u_r = \frac{1 + a}{a + 1} = (1 + a) \times \frac{b}{a + 1} = b \]

Therefore, the sequence is periodic and order 5

\[ b \quad \text{When } a = 2 \text{ and } b = 9 \]
\[ u_1 = 2 \]
\[ u_2 = 9 \]
\[ u_3 = \frac{1 + 9}{2} = 5 \]
\[ u_4 = \frac{2 + 9 + 1}{2 \times 9} = \frac{2}{3} \]
\[ u_5 = \frac{2 + 1}{9} = \frac{1}{3} \]
\[ \sum_{r=1}^{5} u_r = 2 + 9 + 5 + \frac{2}{3} + \frac{1}{3} = 17 \]

Series is periodic so \[ \sum_{r=1}^{10} u_r = \sum_{r=6}^{10} u_r = \sum_{r=11}^{15} u_r \]

and so on.

\[ c \quad \text{So } \sum_{r=1}^{100} u_r = 20 \times \sum_{r=1}^{5} u_r = 20 \times 17 = 340 \]