Sequences and series 3F

1 a i $\sum_{r=1}^{5}(3r+1) = 4+7+10+13+16$

ii $S_5 = 50$

b i $\sum_{r=1}^{6}3r^2 = 3+12+27+48+75+108$

ii $S_6 = 273$

c i $\sum_{r=1}^{5}\sin(90r°) = 1+0+(-1)+0+1$

ii $S_5 = 1$

d i $\sum_{r=5}^{8}2\left(-\frac{1}{3}\right)^r = -\frac{2}{243}+\frac{2}{729}-\frac{2}{2187}+\frac{2}{6561}$

ii $S_4 = -\frac{40}{6561}$

2 a i $2+4+6+8 = \sum_{r=1}^{4}2r$

ii $S_4 = 20$

b i $2+6+18+54+162 = \sum_{r=1}^{5}\left(2\times 3^{r-1}\right)$

ii $S_5 = 242$

c i

$6+4.5+3+1.5+0-1.5 = \sum_{r=1}^{6}\left(-\frac{3}{2}r+\frac{15}{2}\right)$

ii $S_6 = 13.5$

3 a i $7+13+19+\ldots+157 = \sum_{r=1}^{n}(6r+1)$

$6n + 1 = 157$
$n = 26$

ii $\sum_{r=1}^{26}(6r+1)$

b i

$\frac{1}{3}+\frac{2}{15}+\frac{4}{75}+\ldots+\frac{64}{46\,875} = \sum_{r=1}^{n}\left(\frac{1}{3}\times\left(\frac{2}{5}\right)^{r-1}\right)$

$\frac{1}{3}\times\left(\frac{2}{5}\right)^{n-1} = \frac{64}{46\,875}$

$\left(\frac{2}{5}\right)^{n-1} = \frac{64}{15\,625}$

$n = \frac{\log(0.004096)}{\log(0.4)}+1 = 7$

ii $\sum_{r=1}^{7}\left(\frac{1}{3}\times\left(\frac{2}{5}\right)^{r-1}\right)$

c i $8-1-10-19-\ldots-127 = \sum_{r=1}^{n}(17-9r)$

$17-9n = -127$
$n = 16$

ii $\sum_{r=1}^{16}(17-9r)$

4 a $\sum_{r=1}^{20}(7-2r) = 5+3+1+\ldots-33$

$a = 5$, $l = -33$, $n = 20$

$S_{20} = \frac{20}{2}(5-33)$

$= -280$

4 b $\sum_{r=1}^{10} 3\times 4^r = 12+48+192+\ldots+3\,145\,728$

$a=12$, $r=4$, $n=10$

$S_{10} = \dfrac{12(4^{10}-1)}{4-1}$

$= 4\,194\,300$

c $\sum_{r=1}^{100}(2r-8) = -6-4-2+\ldots+192$

$a=-6$, $l=192$, $n=100$

$S_{100} = \dfrac{100}{2}(-6+192)$

$= 9300$

d $\sum_{r=1}^{\infty} 7\left(-\dfrac{1}{3}\right)^r = -\dfrac{7}{3}+\dfrac{7}{9}-\dfrac{7}{27}+\ldots$

$a=-\dfrac{7}{3}$, $r=-\dfrac{1}{3}$

$S_\infty = \dfrac{-\dfrac{7}{3}}{1+\dfrac{1}{3}}$

$= -\dfrac{7}{4}$

5 a $\sum_{r=9}^{30}\left(5r-\dfrac{1}{2}\right) = 44\dfrac{1}{2}+49\dfrac{1}{2}+\ldots+149\dfrac{1}{2}$

$a=44\dfrac{1}{2}$, $l=149\dfrac{1}{2}$, $n=22$

$S_{22} = \dfrac{22}{2}\left(44\dfrac{1}{2}+149\dfrac{1}{2}\right)$

$= 2134$

b $\sum_{r=100}^{200}(3r+4) = 304+307+310+\ldots+604$

$a=304$, $l=604$, $n=101$

$S_{101} = \dfrac{101}{2}(304+604)$

$= 45\,854$

c

$\sum_{r=5}^{100} 3\times 0.5^r = 0.09375+0.046875+0.0234375+\ldots$

$a=0.09375$, $r=0.5$, $n=96$

$S_{96} = \dfrac{0.09375(1-0.5^{96})}{1-0.5}$

$= 0.1875$

d $\sum_{i=5}^{100} 1 = 1+1+1+\ldots+1$

$a=1$, $l=1$, $n=96$

$S_{96} = \dfrac{96}{2}(1+1)$

$= 96$

These are the answers to Q6 and Q7 in the 2020 update to the student book. The answers to the original questions are below.

6 $\sum_{r=1}^{30}(r+2^r) = \sum_{r=1}^{30} r + \sum_{r=1}^{30} 2^r$

$\sum_{r=1}^{30} r = \dfrac{30(30+1)}{2} = 465$

$\sum_{r=1}^{30} 2^r = 2+4+8+\ldots$

$a=2$, $r=2$, $n=30$

$S_n = \dfrac{a(r^n-1)}{r-1}$

$S_{30} = \dfrac{2(2^{30}-1)}{2-1} = 2147\,483\,646$

So $\sum_{r=1}^{30}(r+2^r) = \sum_{r=1}^{30} r + \sum_{r=1}^{30} 2^r$

$= 465 + 2147\,483\,646$

$= 2147\,484\,111$

7 $\sum_{r=1}^{12}(2r-5+3^r) = \sum_{r=1}^{12}(2r-5) + \sum_{r=1}^{12} 3^r$

$\sum_{r=1}^{12}(2r-5) = (-3)+(-1)+1+\ldots+19$

$\boldsymbol{a=-3, d=2, n=12}$

$S_n = \dfrac{n}{2}(2a+(n-1)d)$

$S_{12} = \dfrac{12}{2}(2\times(-3)+(12-1)(2)) = 96$

$\sum_{r=1}^{12} 3^r = 3+9+27+\ldots$

$\boldsymbol{a = 3, r = 3, n = 12}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{12} = \frac{3(3^{12} - 1)}{3 - 1} = 797160$$

So

$$\sum_{r=1}^{12}(2r - 5 + 3^r) = \sum_{r=1}^{12}(2r - 5) + \sum_{r=1}^{12}3^r = 96 + 797160 = 797256$$

These are the answers to Q6 and Q7 in the original version of the student book.

6 $\sum_{r=1}^{n} 2r = 2 + 4 + 6 + \ldots + 2n$

$a = 2,\ l = 2n$

$$S_n = \frac{n}{2}(2 + 2n)$$

$$= n + n^2$$

7 $\sum_{r=1}^{n} 2r = n + n^2$

$\sum_{r=1}^{n}(2r - 1) = 1 + 3 + 5 + \ldots + (2n - 1)$

$a = 1,\ l = 2n - 1$

$$S_n = \frac{n}{2}(1 + 2n - 1)$$

$$= n^2$$

$$\sum_{r=1}^{n} 2r - \sum_{r=1}^{n}(2r - 1) = n + n^2 - n^2 = n$$

8 a $\sum_{r=1}^{k} 4(-2)^r = -8 + 16 + \ldots + 4(-2)^k$

$a = -8, r = -2$

$$S_k = \frac{-8(1 - (-2)^k)}{1 + 2}$$

$$= \frac{8}{3}\left((-2)^k - 1\right)$$

b $\sum_{r=1}^{k}(100 - 2r) = 98 + 96 + \ldots + (100 - 2k)$

$a = 98,\ l = 100 - 2k$

$$S_k = \frac{k}{2}(98 + 100 - 2k)$$

$$= 99k - k^2$$

8 c $\sum_{r=10}^{k}(7 - 2r) = -13 - 15 - \ldots + (7 - 2k)$

$a = -13,\ l = 7 - 2k$

$$S_{k-9} = \frac{k - 9}{2}(-13 + 7 - 2k)$$

$$= (k - 9)(-3 - k)$$

$$= 6k - k^2 + 27$$

9 $\sum_{r=10}^{\infty} 200 \times \left(\frac{1}{4}\right)^r =$

$$\sum_{r=1}^{\infty} 200 \times \left(\frac{1}{4}\right)^r - \sum_{r=1}^{r=9} 200 \times \left(\frac{1}{4}\right)^r$$

$a = 50,\ r = \frac{1}{4}$

$$S_\infty - S_9 = \frac{50}{1 - \frac{1}{4}} - \frac{50\left(1 - \left(\frac{1}{4}\right)^9\right)}{1 - \frac{1}{4}}$$

$$= \frac{50 - 50\left(1 - \left(\frac{1}{4}\right)^9\right)}{\frac{3}{4}}$$

$$= \frac{200}{3}\left(\frac{1}{4}\right)^9$$

$$= \frac{25}{98\,304}$$

10 a $\sum_{r=1}^{k}(8 + 3r) = 11 + 14 + 17 + \ldots + (8 + 3k)$

$a = 11,\ l = 8 + 3k,\ n = k$

$$S_k = \frac{k}{2}(11 + 8 + 3k)$$

$$\frac{k}{2}(19 + 3k) = 377$$

$$19k + 3k^2 = 754$$

$$3k^2 + 19k - 754 = 0$$

$$(3k + 58)(k - 13) = 0$$

b As $k > 0$, $k = 13$

11 a $\sum_{r=1}^{k} 2\times 3^r = 59\,046$

$a = 6,\ r = 3$

$S_k = \frac{6(3^k - 1)}{3-1} = 59\,046$

$3(3^k - 1) = 59\,046$

$3^k = 19\,683$

$k\log 3 = \log 19\,683$

$k = \frac{\log 19\,683}{\log 3}$

b $k = 9$

$\sum_{r=10}^{13} 2\times 3^r = \sum_{r=1}^{13} 2\times 3^r - \sum_{r=1}^{9} 2\times 3^r$

$= \frac{6(3^{13} - 1)}{3-1} - 59\,046$

$= 4\,782\,966 - 59\,046$

$= 4\,723\,920$

12 a $r = 3x$

As the series is convergent, $|3x| < 1$

$|x| < \frac{1}{3}$

b $S_\infty = \frac{1}{1-3x} = 2$

$1 = 2(1-3x)$

$6x = 1$

$x = \frac{1}{6}$

Challenge

$\sum_{r=1}^{10} (a+(r-1)d) = a + (a+d) + \ldots + (a+9d)$

$= \frac{10}{2}(a+a+9d)$

$= 5(2a+9d)$

$\sum_{r=11}^{14} (a+(r-1)d) = (a+10d) + \ldots + (a+13d)$

$= \frac{4}{2}(a+10d+a+13d)$

$= 2(2a+23d)$

As $\sum_{r=1}^{10} (a+(r-1)d) = \sum_{r=11}^{14} (a+(r-1)d)$

$5(2a+9d) = 2(2a+23d)$

$10a + 45d = 4a + 46d$

$d = 6a$