Sequences and series 3E

1 a i $r = 0.1$ so the series is convergent as $|r| < 1$.
ii $S_n = \frac{1}{1 - 0.1} = \frac{10}{9}$

b $r = 2$ so the series is not convergent as $|r| \geq 1$.

c i $r = -0.5$ so the series is convergent as $|r| < 1$.
ii $S_n = \frac{10}{1 + 0.5} = \frac{20}{3} = 6 \frac{2}{3}$

d This is an arithmetic series and so does not converge.

e $r = 1$ so the series is not convergent as $|r| \geq 1$.

f i $r = \frac{1}{3}$ so the series is convergent as $|r| < 1$.
ii $S_n = \frac{3}{1 - \frac{1}{3}} = \frac{9}{2} = 4 \frac{1}{2}$

g This is an arithmetic series and so does not converge.

h i $r = 0.9$ so the series is convergent as $|r| < 1$.
ii $S_n = \frac{9}{1 - 0.9} = 90$
6 \ \ \ \ 0.23\ldots = \frac{23}{100} + \frac{23}{10000} + \frac{23}{1000000} + \ldots \\
= \frac{1}{100} \times 0.23 + \frac{1}{100} \times 0.023 + \frac{1}{100} \times 0.0023 + \ldots \\
This is an infinite geometric series: 
\[ a = \frac{23}{100} \quad \text{and} \quad r = \frac{1}{100}. \]

Use \( S_{\infty} = \frac{a}{1-r} \).

\[ S_{\infty} = \frac{23}{100} \times \frac{99}{99} = \frac{23}{100} \times 99 = \frac{23}{100} \]

7 \ \ \ \ S_3 = 9, \ \ \ \ S_{\infty} = 8 \\
S_3 = \frac{a(1-r^3)}{1-r} = 9 \quad (1) \\
S_\infty = \frac{a}{1-r} = 8 \quad (2) \\
8(1-r^3) = 9 \quad \text{substituting (2) into (1)} \\
1-r^3 = \frac{9}{8} \\
r^3 = -\frac{1}{8} \\
r = -\frac{1}{2} \\
a = 8\left(1 + \frac{1}{2}\right) \quad \text{(from (2))} \\
a = 12 \\

8 \ a \ a = 1, \ r = -2x \\
As the series is convergent, \(-2x \times 1\) \\
If \( x < 0 \) then \( 2x < 1 \), so \( x < \frac{1}{2} \); \\
if \( x > 0 \) then \(-2x < 1 \), so \( x > -\frac{1}{2} \) \\
Hence, \(-\frac{1}{2} < x < \frac{1}{2}\).

b \ \ \ \ S_\infty = \frac{1}{1+2x}

9 \ a \ a = 2, \ S_\infty = 16 \times S_3 \\
S_3 = \frac{2(1-r^3)}{1-r} \\
16 \times \frac{2(1-r^3)}{1-r} = \frac{2}{1-r} \\
32(1-r^3) = 2 \\
r^3 = \frac{15}{16} \\
r = 0.9787 \\

b \ \ \ u_4 = ar^2 = 2 \times 0.9787^3 = 1.875

10 \ a \ a = 30, \ S_\infty = 240 \\
\frac{30}{1-r} = 240 \\
\frac{1}{8} = 1-r \\
r = \frac{7}{8} \\

b \ \ \ u_4 - u_5 = ar^3 - ar^4 \\
= 30\left(\frac{7}{8}\right)^3 - 30\left(\frac{7}{8}\right)^4 \\
= 2.51 \\

30\left(1 - \left(\frac{7}{8}\right)^4\right) \\

\ \ \ \ \ \ \ \ \ = \frac{99.3}{7/8} \\

\ \ \ \ \ \ \ \ \ = 99.3
10 d If \( S_n = \frac{30 \left(1 - \left(\frac{7}{8}\right)^n\right)}{1 - \frac{7}{8}} = 180 \),

\[ \frac{30 \left(1 - \left(\frac{7}{8}\right)^n\right)}{1 - \frac{7}{8}} = 180 \]

\[ 1 - \left(\frac{7}{8}\right)^n = 0.75 \]

0.875\(^n\) = 0.25

\[ n = \frac{\log 0.25}{\log 0.875} \]

\[ n = 10.38 \]

\[ n = 11 \]

11 a \( ar = \frac{15}{8} \), \( S_\infty = 8 \)

\[ a \frac{1}{1-r} = 8 \]

\[ a = 8(1 - r) \]

\[ a = \frac{15}{8r} \]

\[ \frac{15}{8r} = 8(1 - r) \]

\[ 15 = 64r - 64r^2 \]

\[ 64r^2 - 64r + 15 = 0 \]

b \( (8r - 3)(8r - 5) = 0 \)

\[ r = \frac{3}{8} \text{ or } r = \frac{5}{8} \]

c When \( r = \frac{3}{8} \),

\[ a = 8 \left(1 - \frac{3}{8}\right) = 5 \]

When \( r = \frac{5}{8} \),

\[ a = 8 \left(1 - \frac{5}{8}\right) = 3 \]

\[ d \quad r = \frac{3}{8} \]

\[ 5 \left(1 - \left(\frac{3}{8}\right)^n\right) = 7.99 \]

\[ \frac{5 \left(1 - \left(\frac{3}{8}\right)^n\right)}{1 - \frac{3}{8}} = 7.99 \]

\[ 1 - 0.375^n = 0.99875 \]

0.375\(^n\) = 0.00125

\[ n = \frac{\log 0.00125}{\log 0.375} \]

\[ n = 6.815 \]

\[ n = 7 \]

**Challenge**

a First series: \( a + ar + ar^2 + ar^3 + \ldots \)

Second series: \( a^2 + a^2r^2 + a^2r^3 + \ldots \)

Second series has first term \( a^2 \) and common ratio \( r^2 \) so is a geometric series.

b For the first series: \( S_\infty = 7 \)

\[ \frac{a}{1-r} = 7 \]

\[ a = 7(1 - r) \]

For the second series: \( S_\infty = 35 \)

\[ \frac{a^2}{1-r^2} = 35 \]

\[ \frac{a^2}{(1-r)(1+r)} = 35 \]

\[ \frac{49(1-r)^2}{(1-r)(1+r)} = 35 \]

\[ 49 - 49r = 35 + 35r \]

\[ 14 = 84r, \text{ so } r = \frac{1}{6} \]