Sequences and series 3D

1 a $1+2+4+8+\ldots$ (8 terms)

In this series $a=1, r=2, n=8$.

As $|r|>1$ use $S_n=\dfrac{a(r^n-1)}{r-1}$.

$$S_8=\frac{a(r^8-1)}{r-1}=\frac{1\times(2^8-1)}{2-1}=256-1=255$$

b $32+16+8+\ldots$ (10 terms)

In this series $a=32, r=\dfrac{1}{2}, n=10$.

As $|r|<1$ use $S_n=\dfrac{a(1-r^n)}{1-r}$.

$$S_{10}=\frac{a(1-r^{10})}{1-r}$$
$$=\frac{32\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}=63.938\text{ (3 d.p.)}$$

c $a=\dfrac{2}{3}, r=\dfrac{2}{5}, n=8$

$$S_8=\frac{\frac{2}{3}\left(1-\left(\frac{2}{5}\right)^8\right)}{1-\frac{2}{5}}$$
$=1.110$

d $4-12+36-108+\ldots$ (6 terms)

In this series $a=4, r=-3, n=6$.

As $|r|>1$ use $S_n=\dfrac{a(r^n-1)}{r-1}$.

$$S_6=\frac{a(r^6-1)}{r-1}=\frac{4\left((-3)^6-1\right)}{-3-1}=-728$$

e $729-243+81-\ \ldots\ -\dfrac{1}{3}$

Here, $a=729, r=\dfrac{-243}{729}=-\dfrac{1}{3}$

and the nth term is $-\dfrac{1}{3}$.

Using nth term $=ar^{n-1}$

$$-\frac{1}{3}=729\times\left(-\frac{1}{3}\right)^{n-1}$$

$$-\frac{1}{2187}=\left(-\frac{1}{3}\right)^{n-1}$$

$$\left(-\frac{1}{3}\right)^7=\left(-\frac{1}{3}\right)^{n-1}$$

So $n-1=7$

$\Rightarrow n=8$

There are 8 terms in the series.

As $|r|<1$ use $S_n=\dfrac{a(1-r^n)}{1-r}$ with $a=729, r=-\dfrac{1}{3}$ and $n=8$.

$$S_8=\frac{729\left(1-\left(-\frac{1}{3}\right)^8\right)}{1-\left(-\frac{1}{3}\right)}=546\frac{2}{3}$$

1 f $a = -\frac{5}{2}, r = -\frac{1}{2}, n = 15$

$$S_{15} = \frac{-\frac{5}{2}\left(1-\left(-\frac{1}{2}\right)^{15}\right)}{1+\frac{1}{2}}$$

$= -1.667$

2 $a = 3, r = 0.4, n = 10$

$$S_{10} = \frac{3\left(1-0.4^{10}\right)}{1-0.4}$$

$= 4.9995$

3 $a = 5, r = \frac{2}{3}, n = 8$

$$S_8 = \frac{5\left(1-\left(\frac{2}{3}\right)^{8}\right)}{1-\frac{2}{3}}$$

$= 14.4147$

4 Let the common ratio be r.

The first three terms are 8, $8r$ and $8r^2$.

Given that the first three terms add up to 30.5,

$$8+8r+8r^2 = 30.5 \quad (\times 2)$$
$$16+16r+16r^2 = 61$$
$$16r^2+16r-45 = 0$$
$$(4r-5)(4r+9) = 0$$
$$r = \frac{5}{4}, \frac{-9}{4}$$

Possible values of r are $\frac{5}{4}$ and $\frac{-9}{4}$.

5 3 + 6 + 12 + 24 + … is a geometric series with $a = 3$, $r = 2$.

So $S_n = \frac{a\left(r^n-1\right)}{r-1} = \frac{3\left(2^n-1\right)}{2-1} = 3\left(2^n-1\right)$

We want $S_n > 1.5$ million

$$S_n > 1500000$$
$$3\left(2^n-1\right) > 1500000$$
$$2^n-1 > 500000$$
$$2^n > 500001$$
$$\log 2^n > \log 500001$$
$$n \log 2 > \log 500001$$
$$n > \frac{\log 500001}{\log 2}$$
$$n > 18.9$$

Least value of n is 19.

6 5 + 4.5 + 4.05 + … is a geometric series with $a = 5$ and $r = \frac{4.5}{5} = 0.9$.

Using $S_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{5\left(1-0.9^n\right)}{1-0.9}$

$= 50\left(1-0.9^n\right)$

We want $S_n > 45$

$$50\left(1-0.9^n\right) > 45$$
$$\left(1-0.9^n\right) > \frac{45}{50}$$
$$1-0.9^n > 0.9$$
$$0.9^n < 0.1$$

6 (continued)

$\log(0.9)^n < \log(0.1)$

$n\log(0.9) < \log(0.1)$

$n > \dfrac{\log(0.1)}{\log(0.9)}$

$n > 21.85$

So $n = 22$

7 a $a = 25,\ r = \dfrac{3}{5},\ S_k > 61$

$\dfrac{25\left(1-\left(\dfrac{3}{5}\right)^k\right)}{1-\dfrac{3}{5}} > 61$

$\dfrac{25\left(1-0.6^k\right)}{0.4} > 61$

$25(1 - 0.6^k) > 24.4$

$1 - 0.6^k > 0.976$

$0.6^k > 0.024$

$k\log(0.6) > \log(0.024)$

$k > \dfrac{\log(0.024)}{\log(0.6)}$

b $k > 7.301$

$k = 8$

8 $S_2 = \dfrac{a\left(1-r^2\right)}{1-r} = 4.48$

$a\left(1-r^2\right) = 4.48(1-r)$

$a = \dfrac{4.48(1-r)}{1-r^2}$

$S_4 = \dfrac{a\left(1-r^4\right)}{1-r} = 5.1968$

$a\left(1-r^4\right) = 5.1968(1-r)$

$a = \dfrac{5.1968(1-r)}{1-r^4}$

$\dfrac{4.48(1-r)}{1-r^2} = \dfrac{5.1968(1-r)}{1-r^4}$

$\dfrac{1}{1-r^2} = \dfrac{1.16}{1-r^4}$

$\dfrac{1}{1-r^2} = \dfrac{1.16}{(1-r^2)(1+r^2)}$

$1 = \dfrac{1.16}{(1+r^2)}$

$1 + r^2 = 1.16$

$r^2 = 0.16$

$r = \pm 0.4$

9 $a = a,\ r = \sqrt{3}$

$S_{10} = \dfrac{a\left(\sqrt{3}^{10}-1\right)}{\sqrt{3}-1}$

$= \dfrac{a(243-1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$= \dfrac{242a(\sqrt{3}+1)}{3-1}$

$= 121a(\sqrt{3}+1)$

10 First series:

$a = a,\ r = 2$

$S_4 = \dfrac{a\left(2^4-1\right)}{2-1}$

$S_4 = 15a$

Second series:

$a = b,\ r = 3$

$S_4 = \dfrac{b\left(3^4-1\right)}{3-1}$

$S_4 = 40b$

$15a = 40b$

$a = \dfrac{8}{3}b$

11 a $\frac{2k+5}{k} = \frac{k}{k-6}$

$(2k + 5)(k - 6) = k^2$

$2k^2 + 7k - 30 = k^2$

$k^2 + 7k - 30 = 0$

b $(k + 3)(k - 10) = 0$

$k = -3$ or $k = 10$

As $k > 0$, $k = 10$

c $r = \frac{10}{10-6} = \frac{5}{2} = 2.5$

d $S_{10} = \frac{4\left(2.5^{10} - 1\right)}{2.5 - 1}$

$= 25\,429$

Sequences and series 3E

1 a i $r = 0.1$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{1}{1-0.1} = \frac{10}{9}$

b $r = 2$ so the series is not convergent as $|r| \geqslant 1$.

c i $r = -0.5$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{10}{1+0.5} = \frac{20}{3} = 6\frac{2}{3}$

d This is an arithmetic series and so does not converge.

e $r = 1$ so the series is not convergent as $|r| \geqslant 1$.

f i $r = \frac{1}{3}$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{3}{1-\frac{1}{3}} = \frac{9}{2} = 4\frac{1}{2}$

g This is an arithmetic series and so does not converge.

h i $r = 0.9$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{9}{1-0.9} = 90$

2 $a = 10,\ S_\infty = 30$

$\frac{10}{1-r} = 30$

$10 = 30(1-r)$

$30r = 20$

$r = \frac{2}{3}$

3 $a = -5,\ S_\infty = -3$

$\frac{-5}{1-r} = -3$

$-5 = -3(1-r)$

$3r = -2$

$r = -\frac{2}{3}$

4 $S_\infty = 60,\ r = \frac{2}{3}$

$\frac{a}{1-\frac{2}{3}} = 60$

$\frac{a}{\frac{1}{3}} = 60$

$a = 20$

5 $S_\infty = 10,\ r = -\frac{1}{3}$

$\frac{a}{1+\frac{1}{3}} = 10$

$\frac{a}{\frac{4}{3}} = 10$

$a = \frac{40}{3} = 13\frac{1}{3}$

6 $0.\dot{2}\dot{3}\ldots = \dfrac{23}{100} + \dfrac{23}{10\,000} + \dfrac{23}{1\,000\,000} + \ldots$

$\times\frac{1}{100}$ $\times\frac{1}{100}$

This is an infinite geometric series:
$a = \dfrac{23}{100}$ and $r = \dfrac{1}{100}$.

Use $S_\infty = \dfrac{a}{1-r}$.

$$0.\dot{2}\dot{3}\ldots = \frac{\frac{23}{100}}{1-\frac{1}{100}} = \frac{\frac{23}{100}}{\frac{99}{100}}$$
$$= \frac{23}{100} \times \frac{100}{99} = \frac{23}{99}$$

7 $S_3 = 9,\ S_\infty = 8$

$S_3 = \dfrac{a(1-r^3)}{1-r} = 9$ (**1**)

$S_\infty = \dfrac{a}{1-r} = 8$ (**2**)

$8(1-r^3) = 9$ (substituting (**2**) into (**1**))

$1 - r^3 = \dfrac{9}{8}$

$r^3 = -\dfrac{1}{8}$

$r = -\dfrac{1}{2}$

$a = 8\left(1 + \dfrac{1}{2}\right)$ (from (**2**))

$a = 12$

8 a $a = 1, r = -2x$

As the series is convergent, $|-2x| < 1$

If $x < 0$ then $2x < 1$, so $x < \dfrac{1}{2}$;

if $x > 0$ then $-2x < 1$, so $x > -\dfrac{1}{2}$

Hence, $-\dfrac{1}{2} < x < \dfrac{1}{2}$.

b $S_\infty = \dfrac{1}{1+2x}$

9 a $a = 2,\ S_\infty = 16 \times S_3$

$S_3 = \dfrac{2(1-r^3)}{1-r}$

$16 \times \dfrac{2(1-r^3)}{1-r} = \dfrac{2}{1-r}$

$32(1-r^3) = 2$

$r^3 = \dfrac{15}{16}$

$r = 0.9787$

b $u_4 = ar^3 = 2 \times 0.9787^3 = 1.875$

10 a $a = 30,\ S_\infty = 240$

$\dfrac{30}{1-r} = 240$

$\dfrac{1}{8} = 1 - r$

$r = \dfrac{7}{8}$

b $u_4 - u_5 = ar^3 - ar^4$

$= 30\left(\dfrac{7}{8}\right)^3 - 30\left(\dfrac{7}{8}\right)^4$

$= 2.51$

c $S_4 = \dfrac{30\left(1 - \left(\frac{7}{8}\right)^4\right)}{1 - \frac{7}{8}}$

$= 99.3$

10 d If $S_n = \dfrac{30\left(1-\left(\frac{7}{8}\right)^n\right)}{1-\frac{7}{8}} = 180$

$$\frac{30\left(1-\left(\frac{7}{8}\right)^n\right)}{\frac{1}{8}} = 180$$

$$1-\left(\frac{7}{8}\right)^n = 0.75$$

$$0.875^n = 0.25$$

$$n = \frac{\log 0.25}{\log 0.875}$$

$$n = 10.38$$

$$n = 11$$

11 a $ar = \dfrac{15}{8}$, $S_\infty = 8$

$$\frac{a}{1-r} = 8$$

$$a = 8(1-r)$$

$$a = \frac{15}{8r}$$

$$\frac{15}{8r} = 8(1-r)$$

$$15 = 64r - 64r^2$$

$$64r^2 - 64r + 15 = 0$$

b $(8r-3)(8r-5) = 0$

$r = \dfrac{3}{8}$ or $r = \dfrac{5}{8}$

c When $r = \dfrac{3}{8}$

$$a = 8\left(1-\frac{3}{8}\right) = 5$$

When $r = \dfrac{5}{8}$

$$a = 8\left(1-\frac{5}{8}\right) = 3$$

d $r = \dfrac{3}{8}$

If $S_n = \dfrac{5\left(1-\left(\frac{3}{8}\right)^n\right)}{1-\frac{3}{8}} = 7.99$

$$\frac{5\left(1-\left(\frac{3}{8}\right)^n\right)}{\frac{5}{8}} = 7.99$$

$$1-0.375^n = 0.99875$$

$$0.375^n = 0.00125$$

$$n = \frac{\log 0.00125}{\log 0.375}$$

$$n = 6.815$$

$$n = 7$$

Challenge

a First series: $a + ar + ar^2 + ar^3 + \ldots$
Second series: $a^2 + a^2r^2 + a^2r^4 + \ldots$
Second series has first term a^2 and common ratio r^2 so is a geometric series.

b For the first series: $S_\infty = 7$

$$\frac{a}{1-r} = 7$$

$$a = 7(1-r)$$

For the second series: $S_\infty = 35$

$$\frac{a^2}{1-r^2} = 35$$

$$\frac{a^2}{(1-r)(1+r)} = 35$$

$$\frac{49(1-r)^2}{(1-r)(1+r)} = 35$$

$$49 - 49r = 35 + 35r$$

$14 = 84r$, so $r = \dfrac{1}{6}$