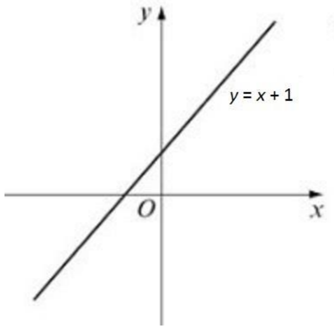
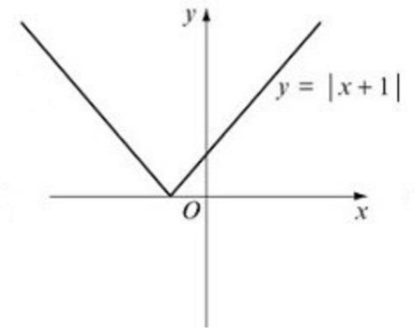
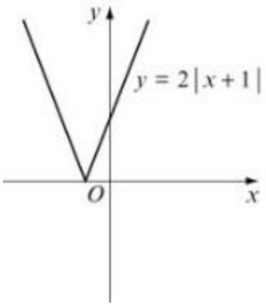
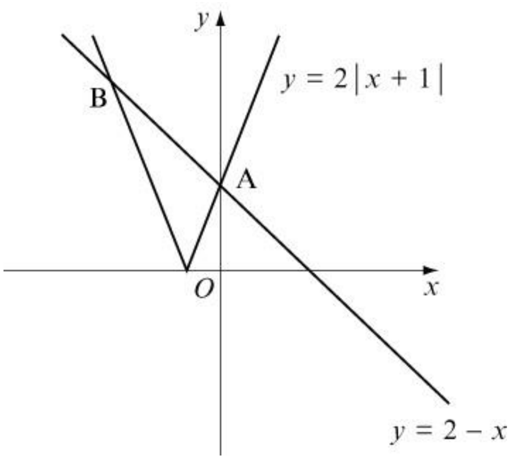
Functions and graphs Mixed exercise 2

1 a









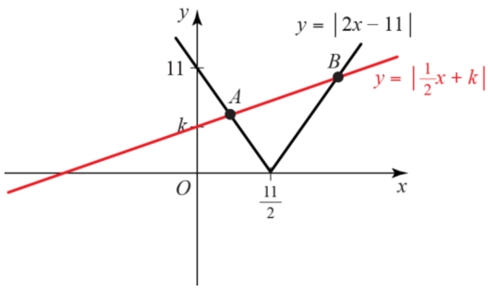
b Intersection point A:

$$2(x+1) = 2 - x$$
$$2x + 2 = 2 - x$$
$$3x = 0$$
$$x = 0$$

Intersection point B is on the reflected part of the modulus graph.

$$-2(x+1) = 2 - x$$
$$-2x - 2 = 2 - x$$
$$-x = 4$$
$$x = -4$$

2



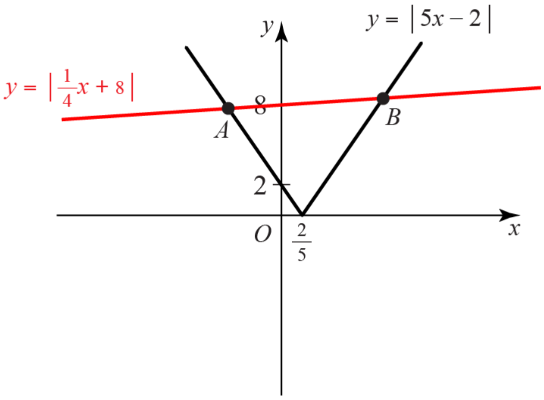
Minimum value of $y = |2x - 11|$ is

$$y = 0 \text{ at } x = \frac{11}{2}$$

For two distinct solutions to $|2x - 11| = \frac{1}{2}x - k$, we must have

$$\frac{1}{2}x - k > 0 \text{ at } x = \frac{11}{2}$$
$$\frac{1}{2} \times \frac{11}{2} + k > 0$$
$$k > -\frac{11}{4}$$

3



At A:

$-(5x-2) = -\frac{1}{4}x + 8$

$-20x + 8 = -x + 32$

$-19x = 24$

$x = -\frac{24}{19}$

At B:

$5x - 2 = -\frac{1}{4}x + 8$

$20x - 8 = -x + 32$

$21x = 40$

$x = \frac{40}{21}$

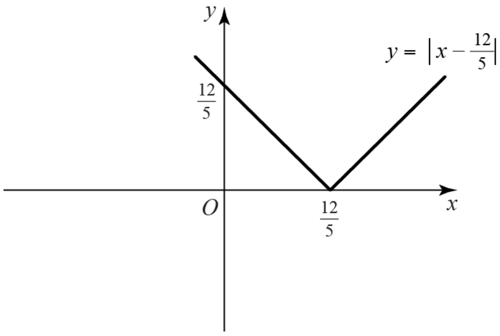
So the solution are

$x = -\frac{24}{19}$ and $x = \frac{40}{21}$

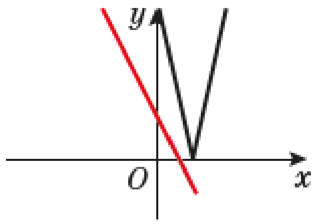
4 a $y = |12 - 5x| = 5\left|-\left(x - \frac{12}{5}\right)\right|$

Start with $y = |x|$

$y = \left|x - \frac{12}{5}\right|$ is a horizontal translation of $+\frac{12}{5}$

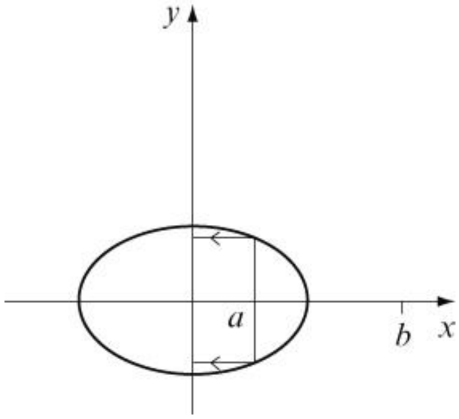


$y = 5\left|x - \frac{12}{5}\right|$ is a vertical stretch, scale factor 5



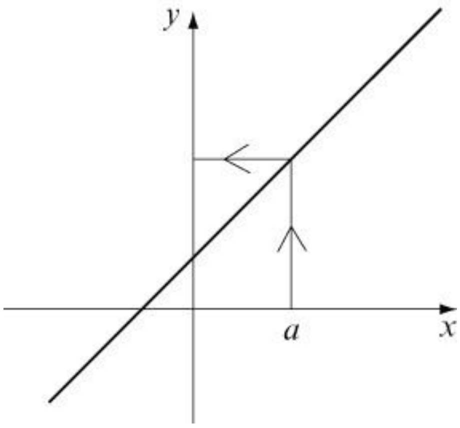
b The graphs do not intersect, so there are no solutions.

5 a i One-to-many.
ii Not a function.

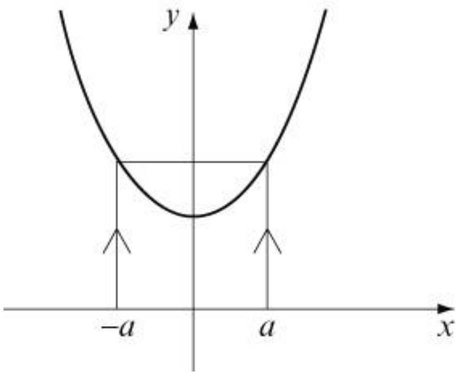


x value a gets mapped to two values of y.
x value b gets mapped to no values.

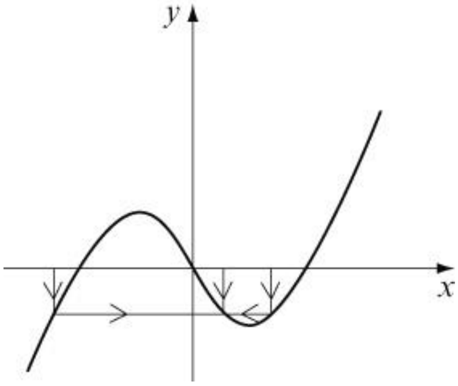
b i One-to-one.
ii Is a function.



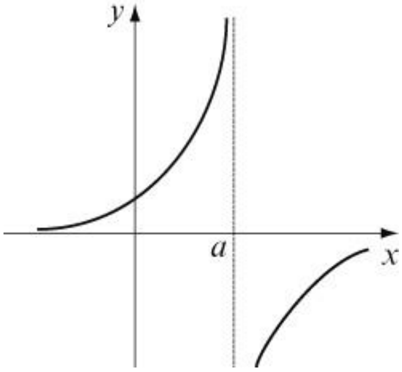
c i Many-to-one.
ii Is a function.



d i Many-to-one.
ii Is a function.

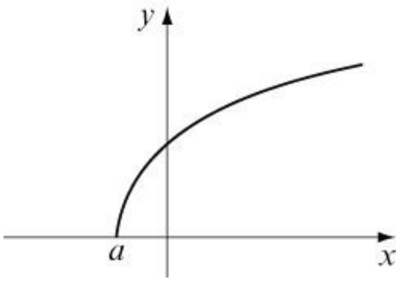


5 e i One-to-one.
ii Not a function.



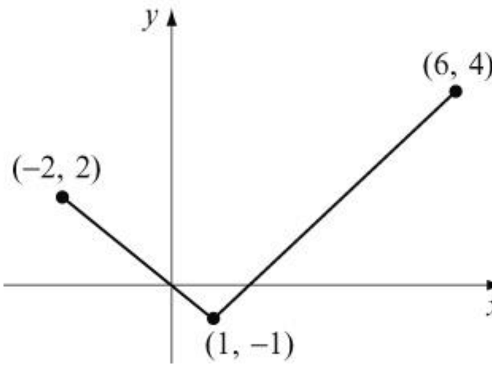
x value a doesn't get mapped to any value of y. It could be redefined as a function if the domain is said to exclude point a.

f i One-to-one.
ii Not a function for this domain.



x values less than a don't get mapped anywhere. Again, we could define the domain to be $x \leqslant a$ and then it would be a function.

6 a



For $x \leqslant 1$, $f(x) = -x$

This is a straight line of gradient -1.

At point $x = 1$, its y-coordinate is -1.

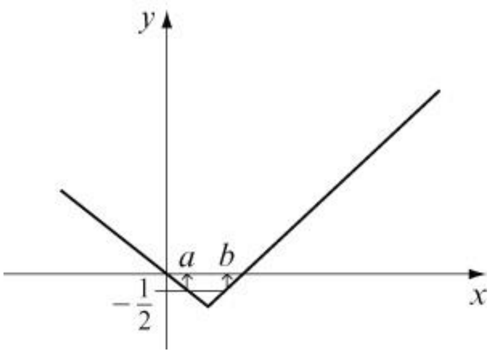
For $x > 1$, $f(x) = x - 2$

This is a straight line of gradient $+1$.

At point $x = 1$, its y-coordinate is also -1.

Hence, the graph is said to be continuous.

b There are two values x in the range $-2 \leqslant x \leqslant 6$ for which $f(x) = -\frac{1}{2}$



Point a is where

$-x = -\frac{1}{2} \Rightarrow x = \frac{1}{2}$

Point b is where

$x - 2 = -\frac{1}{2} \Rightarrow x = 1\frac{1}{2}$

Hence, the values of x for which $f(x) = -\frac{1}{2}$ are $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$

7 a $pq(x) = p(2x + 1)$

$= (2x + 1)^2 + 3(2x + 1) - 4$

$= 4x^2 + 4x + 1 + 6x + 3 - 4$

$= 4x^2 + 10x, \; x \in \mathbb{R}$

b $qq(x) = q(2x + 1)$

$= 2(2x + 1) + 1$

$= 4x + 3$

$pq(x) = qq(x)$ gives

$4x^2 + 10x = 4x + 3$

$4x^2 + 6x - 3 = 0$

Using the formula:

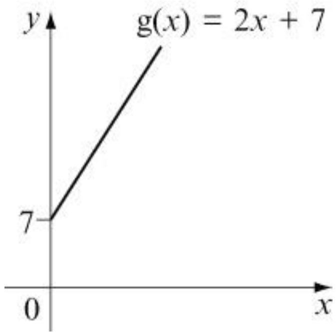
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 4 \times (-3)}}{2 \times 4}$$

$$x = \frac{-6 \pm \sqrt{84}}{8}$$

$$x = \frac{-6 \pm 2\sqrt{21}}{8}$$

$$x = \frac{-3 \pm \sqrt{21}}{4}$$

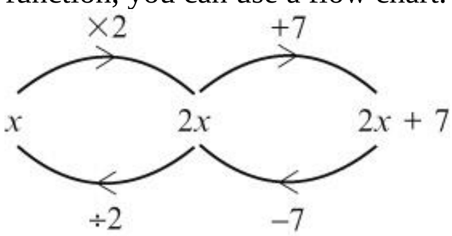
8 a $y = 2x + 7$ is a straight line with gradient 2 and y-intercept 7



For $x \geqslant 0$, the range is $g(x) \geqslant 7$

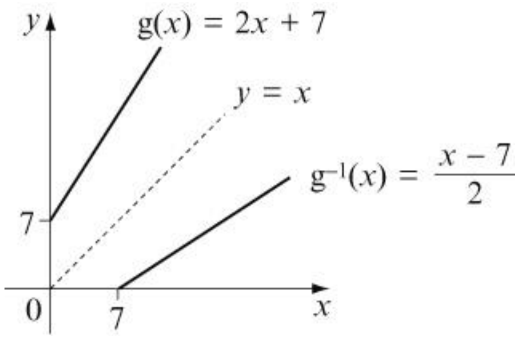
b The range is $g^{-1}(x) \geqslant 0$.

To find the equation of the inverse function, you can use a flow chart.



$g^{-1}(x)=\frac{x-7}{2}$ and has domain $x \geqslant 7$

8 c



$g^{-1}(x)$ is the reflection of $g(x)$ in the line $y=x$.

9 a To find $f^{-1}(x)$, you can change the subject of the formula.

$$\text{Let } y=\frac{2x+3}{x-1}$$
$$y(x-1)=2x+3$$
$$yx-y=2x+3$$
$$yx-2x=y+3$$
$$x(y-2)=y+3$$
$$x=\frac{y+3}{y-2}$$

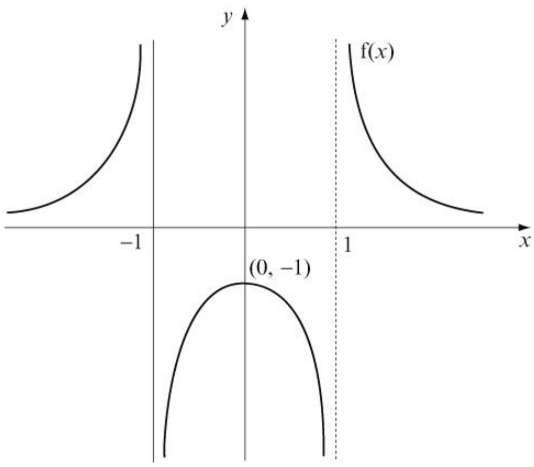
Therefore $f^{-1}(x)=\frac{x+3}{x-2}, x\in\mathbb{R}, x>2$

b Domain $f(x)=$ Range $f^{-1}(x)$

$\therefore$ Range $f^{-1}(x)=\{y\in\mathbb{R}, y>1\}$

10 a
$$f(x)=\frac{x}{x^2-1}-\frac{1}{x+1}$$
$$=\frac{x}{(x+1)(x-1)}-\frac{1}{(x+1)}$$
$$=\frac{x}{(x+1)(x-1)}-\frac{x-1}{(x+1)(x-1)}$$
$$=\frac{x-(x-1)}{(x+1)(x-1)}$$
$$=\frac{1}{(x+1)(x-1)}$$

b Consider the graph of $y=\frac{1}{(x-1)(x+1)}$ for $x\in\mathbb{R}$:



For $x>1, f(x)>0$

c
$$gf(x)=g\left(\frac{1}{(x-1)(x+1)}\right)$$
$$=\frac{2}{\left(\frac{1}{(x-1)(x+1)}\right)}$$
$$=2\times\frac{(x-1)(x+1)}{1}$$
$$=2(x-1)(x+1)$$

$$gf(x)=70\Rightarrow 2(x-1)(x+1)=70$$
$$(x-1)(x+1)=35$$
$$x^2-1=35$$
$$x^2=36$$
$$x=6$$

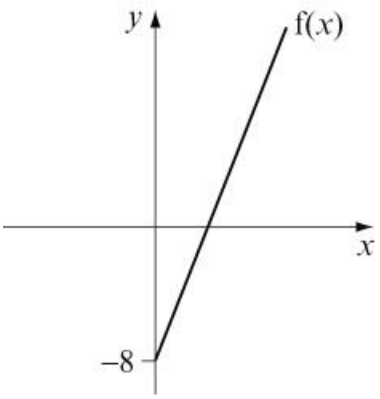
11 a $f(7) = 4(7-2)$
$= 4\times 5$
$= 20$

$g(3) = 3^3 + 1$
$= 27 + 1$
$= 28$

$h(-2) = 3^{-2}$
$= \frac{1}{3^2}$
$= \frac{1}{9}$

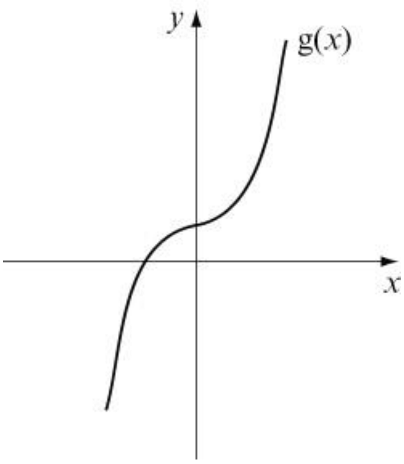
b $f(x) = 4(x-2) = 4x - 8$

This is a straight line with gradient 4 and intercept −8.
The domain tells us that $x \geqslant 0$, so the graph of $y = f(x)$ is:



The range of $f(x)$ is
$f(x) \in \mathbb{R}, f(x) \geqslant -8$

$g(x) = x^3 + 1$



The range of $g(x)$ is $g(x) \in \mathbb{R}$

c Let $y = x^3 + 1$
(change the subject of the formula)

$y - 1 = x^3$
$\sqrt[3]{y-1} = x$

Hence $g^{-1}(x) = \sqrt[3]{x-1}$ $\{x \in \mathbb{R}\}$

d $fg(x) = f(x^3 + 1)$
$= 4(x^3 + 1 - 2)$
$= 4(x^3 - 1), x \in \mathbb{R}, x \geqslant -1$

e First find $gh(x)$:

$gh(x) = g(3^x)$
$= (3^x)^3 + 1$
$= 3^{3x} + 1$

$gh(a) = 244$
$3^{3a} + 1 = 244$
$3^{3a} = 243$
$3^{3a} = 3^5$
$3a = 5$
$a = \frac{5}{3}$

f First find $f^{-1}(x)$
Let $y = 4(x - 2)$
(changing the subject of the formula)

$\frac{y}{4} = x - 2$
$\frac{y}{4} + 2 = x$

Hence $f^{-1}(x) = \frac{x}{4} + 2$

$f^{-1}(x) = -\frac{1}{2}$
$\frac{x}{4} + 2 = -\frac{1}{2}$
$x = 4\left(-\frac{1}{2} - 2\right) = -10$

12 a f^{-1} exists when f is one-to-one.

Now $\text{f}(x) = x^2 + 6x - 4$
Completing the square:
$\text{f}(x) = (x + 3)^2 - 13$
The minimum value is
$\text{f}(x) = -13$ when $x + 3 = 0$

$\Rightarrow x = -3$

Hence, f is one-to-one when $x > -3$
So least value of a is $a = -3$

b Let $y = \text{f}(x)$

$y = x^2 + 6x - 4$
$y = (x + 3)^2 - 13$
$y + 13 = (x + 3)^2$
$x + 3 = \sqrt{y + 13}$
$x = \sqrt{y + 13} - 3$

So $\text{f}^{-1} : x \mapsto \sqrt{x + 13} - 3$

For $a = 0$, Range $\text{f}(x)$ is $y > -4$
So Domain $\text{f}^{-1}(x)$ is $x > -4$

c If $2\text{f}(x) = \text{g}(x)$

$$2 \times (4x - 1) = \frac{3}{2x - 1}$$
$$8x - 2 = \frac{3}{2x - 1}$$
$$(8x - 2)(2x - 1) = 3$$
$$16x^2 - 12x + 2 = 3$$
$$16x^2 - 12x - 1 = 0$$

Use $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

with $a = 16$, $b = -12$ and $c = -1$.

$$\text{Then } x = \frac{12 \pm \sqrt{144 + 64}}{32}$$
$$= \frac{12 \pm \sqrt{208}}{32}$$
$$= 0.826, -0.076$$

Values of x are -0.076 and 0.826

13 a $\text{f} : x \mapsto 4x - 1$

Let $y = 4x - 1$ and change the subject of the formula.

$$\Rightarrow y + 1 = 4x$$
$$\Rightarrow x = \frac{y + 1}{4}$$

Hence $\text{f}^{-1} : x \mapsto \dfrac{x + 1}{4}$, $x \in \mathbb{R}$

b $\text{gf}(x) = \text{g}(4x - 1)$

$$= \frac{3}{2(4x - 1) - 1}$$
$$= \frac{3}{8x - 3}$$

Hence $\text{gf} : x \mapsto \dfrac{3}{8x - 3}$

$\text{gf}(x)$ is undefined when $8x - 3 = 0$

That is, at $x = \dfrac{3}{8}$

$\therefore$ Domain $\text{gf}(x) = \left\{x \in \mathbb{R}, x \neq \dfrac{3}{8}\right\}$

14 a Let $y = \dfrac{x}{x - 2}$

$$y(x - 2) = x$$
$$yx - 2y = x \quad \text{(rearrange)}$$
$$yx - x = 2y$$
$$x(y - 1) = 2y$$
$$x = \frac{2y}{y - 1}$$
$$\text{f}^{-1}(x) = \frac{2x}{x - 1}, \quad x \neq 1$$

b The range of $\text{f}^{-1}(x)$ is the domain of $\text{f}(x)$:

$\{\text{f}^{-1}(x) \in \mathbb{R},\ \text{f}^{-1}(x) \neq 2\}$

c $\text{gf}(1.5) = \text{g}\left(\dfrac{1.5}{1.5 - 2}\right)$

$$= \text{g}\left(\frac{1.5}{-0.5}\right)$$
$$= \text{g}(-3)$$
$$= \frac{3}{-3}$$
$$= -1$$

14 d If $\mathrm{g}(x) = \mathrm{f}(x) + 4$

$$\frac{3}{x} = \frac{x}{x-2} + 4$$

$$3(x-2) = x^2 + 4x(x-2)$$

$$3x - 6 = x^2 + 4x^2 - 8x$$

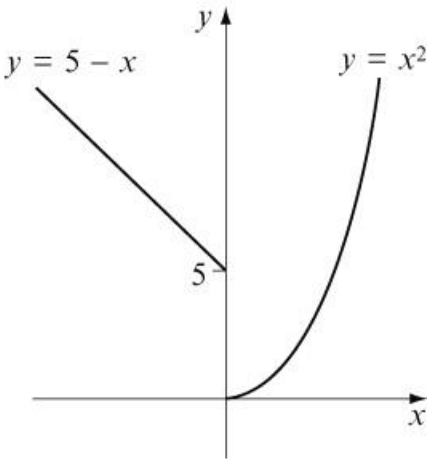
$$0 = 5x^2 - 11x + 6$$

$$0 = (5x-6)(x-1)$$

$$\Rightarrow x = \frac{6}{5}, 1$$

15 $y = 5 - x$ is a straight line with gradient -1 passing through 5 on the y axis.

$y = x^2$ is a $\cup$-shaped quadratic passing through $(0, 0)$



a $\mathrm{n}(-3) = 5 - (-3)$

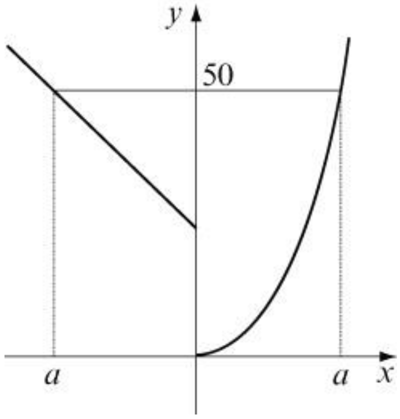
$= 5 + 3$

$= 8$

$\mathrm{n}(3) = 32$

$= 9$

b From the diagram, you can see there are two values of x for which $\mathrm{n}(x) = 50$



The negative value of x is where

$5 - x = 50$

$x = 5 - 50$

$x = -45$

The positive value of x is where

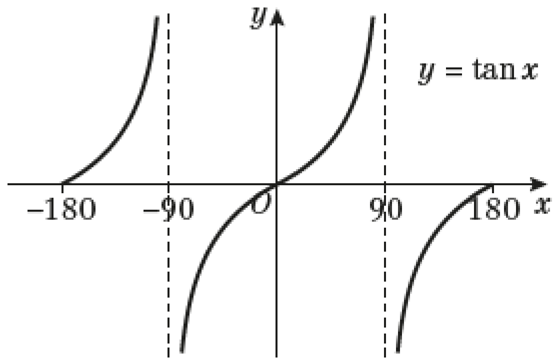
$x^2 = 50$

$x = \sqrt{50}$

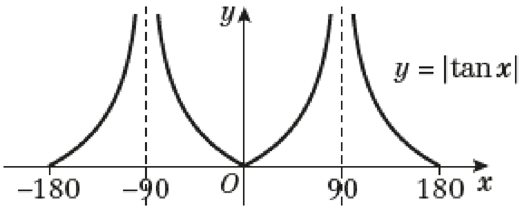
$x = 5\sqrt{2}$

The values of x such that $\mathrm{n}(x) = 50$ are -45 and $+5\sqrt{2}$

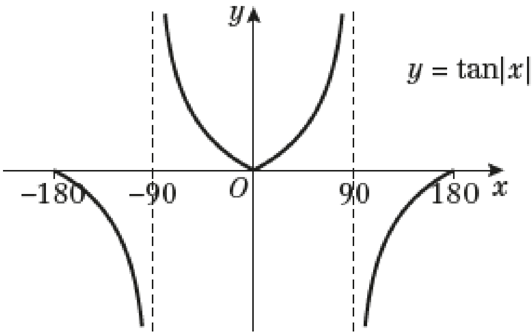
16 a



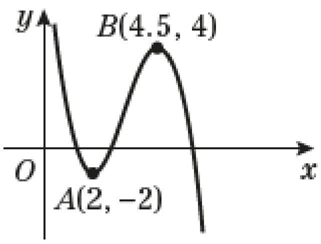
16 b $y=|\tan(x)|$ reflects the negative parts of $\tan x$ in the x axis.



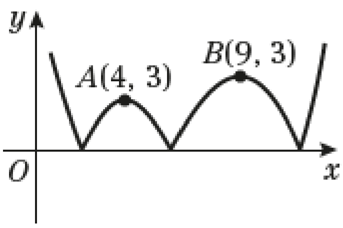
c $y=\tan(|x|)$ reflects $\tan x$ in the y-axis.



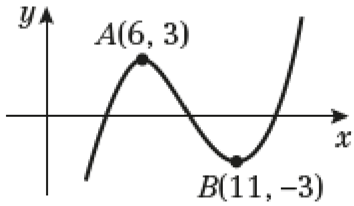
17 a



b



c

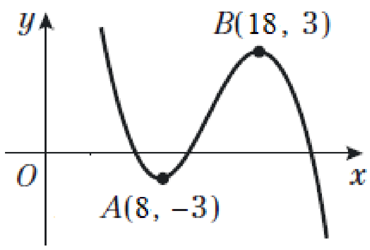


d $y=\mathrm{f}(\frac{1}{2}x+2)$ can be written as

$y=\mathrm{f}(\frac{1}{2}(x+4))$

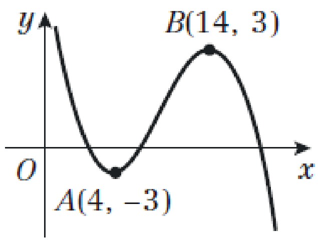
$y=\mathrm{f}(\frac{1}{2}x)$

Horizontal stretch, scale factor 2.



$y=\mathrm{f}(\frac{1}{2}(x+4))$

Horizontal translation of −4

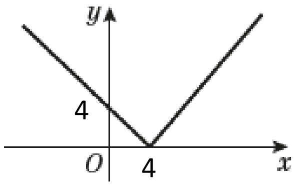


18 a $\mathrm{g}(x)\geqslant 0$

b $\mathrm{gf}(x) = \mathrm{g}(4-x)$
$= 3(4-x)^2$
$= 3x^2 - 24x + 48$

$\mathrm{gf}(x) = 48$
$3x^2 - 24x + 48 = 48$
$3x^2 - 24x = 0$
$3x(x-8) = 0$
$x = 0$ or $x = 8$

18 c

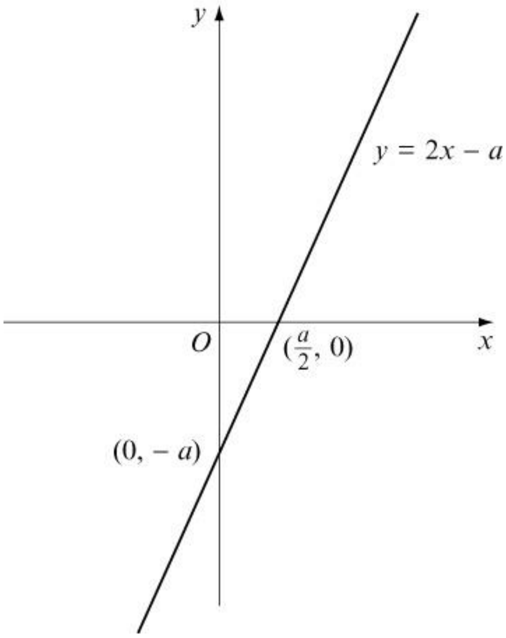


$|f(x)| = 2$ when $|4 - x| = 2$, so

$4 - x = 2 \Rightarrow x = 2$

or $-(4 - x) = 2 \Rightarrow x = 6$

19 a

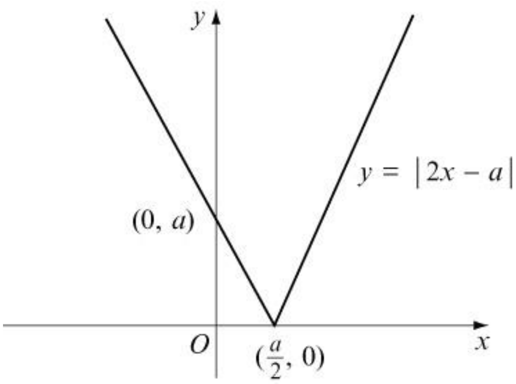


For $y = |2x - a|$:

When $x = 0$, $y = |-a| = a$ $(0, a)$

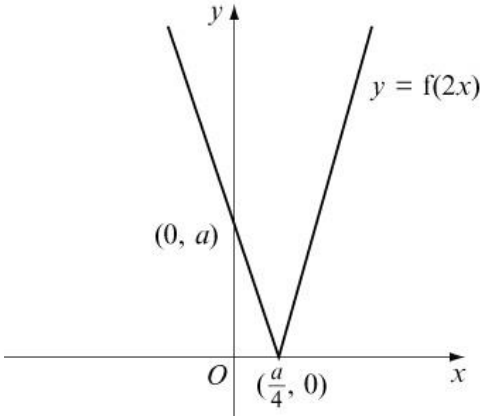
When $y = 0$, $2x - a = 0$

$\Rightarrow x = \frac{a}{2}$ $\left(\frac{a}{2}, 0\right)$



19 b $y = f(2x)$

Horizontal stretch, scale factor $\frac{1}{2}$



c $|2x - a| = \frac{1}{2}x$

Either $(2x - a) = \frac{1}{2}x$

$\Rightarrow a = \frac{3}{2}x$

Given that $x = 4$,

$a = \frac{3 \times 4}{2} = 6$

Or

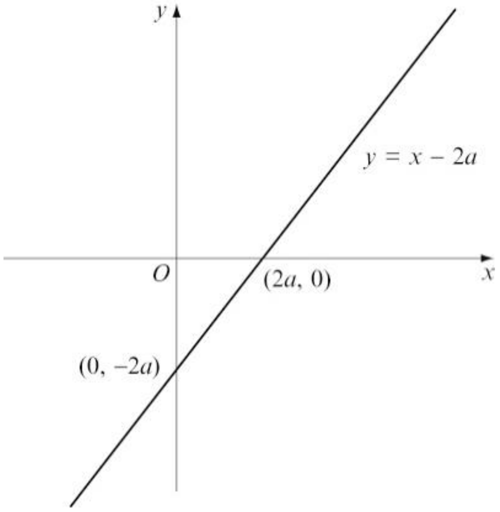
$-(2x - a) = \frac{1}{2}x$

$\Rightarrow a = \frac{5}{2}x$

Given that $x = 4$,

$a = \frac{5 \times 4}{2} = 10$

20 a

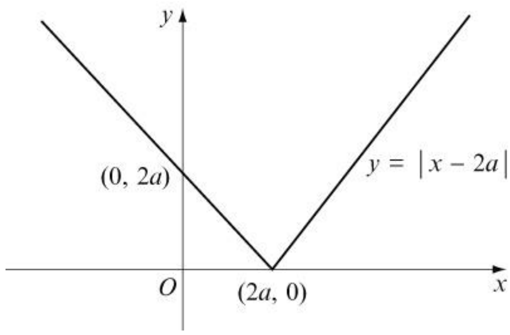


For $y = |x - 2a|$:

When $x = 0$, $y = |-2a| = 2a$ $\quad (0, 2a)$

When $y = 0$, $x - 2a = 0$

$\Rightarrow x = 2a \quad (2a, 0)$



b $|x - 2a| = \frac{1}{3}x$

Either $(x - 2a) = \frac{1}{3}x$

$\Rightarrow x - \frac{1}{3}x = 2a$

$\Rightarrow \frac{2}{3}x = 2a$

$\Rightarrow x = 3a$

or $-(x - 2a) = \frac{1}{3}x$

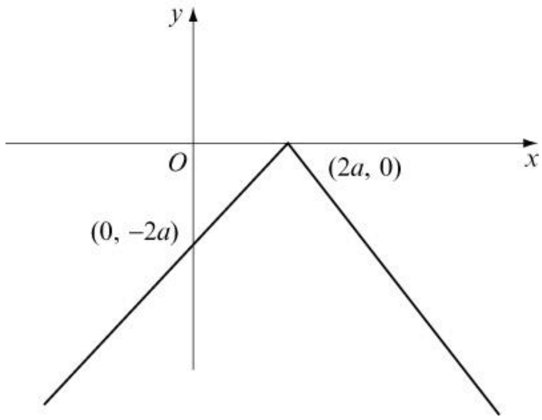
$\Rightarrow -x + 2a = \frac{1}{3}x$

$\Rightarrow \frac{4}{3}x = 2a$

$\Rightarrow x = \frac{3}{2}a$

c $y = -|x - 2a|$

Reflect $y = |x - 2a|$ in the x-axis



$y = a - |x - 2a|$ Vertical translation by $+a$

For $y = a - |x - 2a|$:

When $x = 0$,

$y = a - |-2a|$

$= a - 2a$

$= -a \quad (0, -a)$

When $y = 0$,

$a - |x - 2a| = 0$

$|x - 2a| = a$

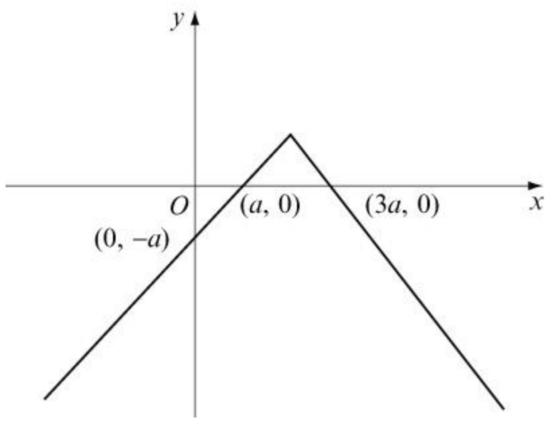
Either $x - 2a = a$

$\Rightarrow x = 3a \quad (3a, 0)$

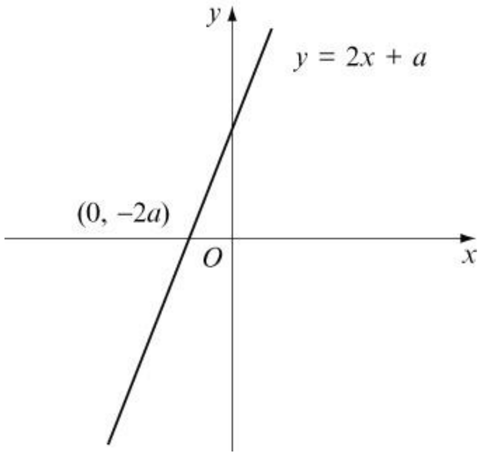
or $-(x - 2a) = a$

$\Rightarrow -x + 2a = a$

$\Rightarrow x = a \quad (a, 0)$



21 a & b

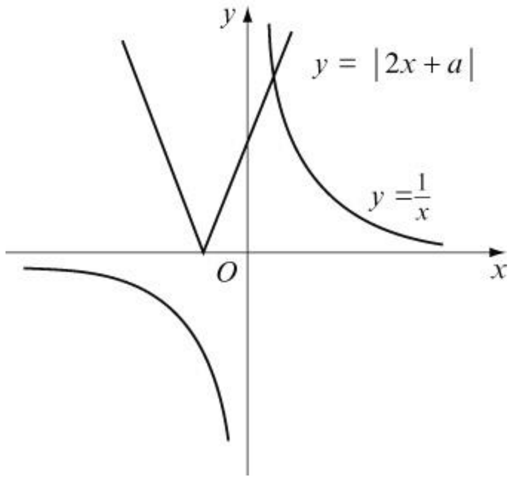


For $y = |2x + a|$:

When $x = 0,\ y = |a| = a$ $\quad (0,\ a)$

When $y = 0,\ 2x + a = 0$

$$\Rightarrow x = -\frac{a}{2} \qquad \left(-\frac{a}{2},\ 0\right)$$



c Intersection of graphs in **b** gives solutions to the equation:

$$|2x + a| = \frac{1}{x}$$

$$x|x + a| = 1$$

$$x|2x + a| - 1 = 0$$

The graphs intersect once only, so $x|2x + a| - 1 = 0$ has only one solution.

d The intersection point is on the non-reflected part of the modulus graph, so here $|2x - a| = 2x - a$

$$x(2x + a) - 1 = 0$$

$$2x^2 + ax - 1 = 0$$

$$x = \frac{-a \pm \sqrt{a^2 + 8}}{4}$$

As shown on the graph, x is positive at intersection,

so $x = \dfrac{-a + \sqrt{a^2 + 8}}{4}$

22 a $f(x) = x^2 - 7x + 5\ln x + 8$

$$f'(x) = 2x - 7 + \frac{5}{x}$$

At stationary points, $f'(x) = 0$:

$$2x - 7 + \frac{5}{x} = 0$$

$$2x^2 - 7x + 5 = 0$$

$$(2x - 5)(x - 1) = 0$$

$$x = \frac{5}{2},\ x = 1$$

Point A: $x = 1$,

$$f(x) = 1 - 7 + 5\ln 1 + 8$$

$$= 2$$

A is $(1,\ 2)$

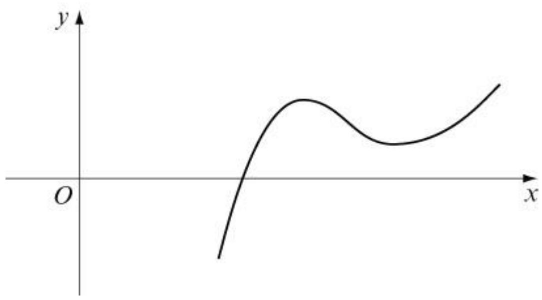
Point B: $x = \dfrac{5}{2}$,

$$f(x) = \frac{25}{4} - \frac{35}{2} + 5\ln\frac{5}{2} + 8$$

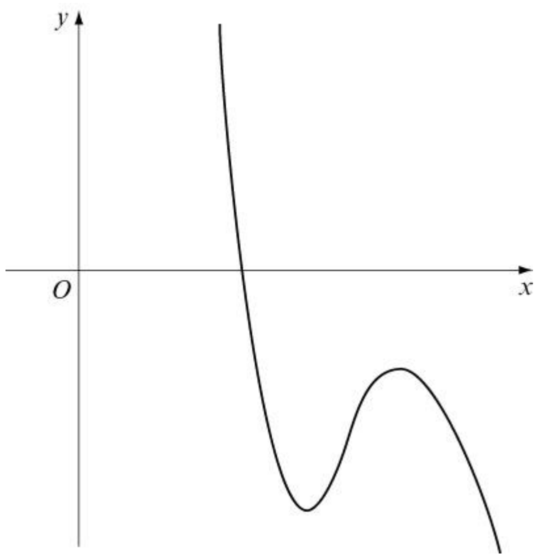
$$= 5\ln\frac{5}{2} - \frac{13}{4}$$

B is $\left(\dfrac{5}{2},\ 5\ln\dfrac{5}{2} - \dfrac{13}{4}\right)$

22 b $y = \mathrm{f}(x-2)$
Horizontal translation of $+2$.
Graph looks like:



$y = -3\mathrm{f}(x-2)$
Reflection in the x-axis, and vertical stretch, scale factor 3.
Graph looks like:



c Using the transformations,
point (X, Y)
becomes $(X+2, -3Y)$

$(1, 2) \to (3, -6)$
Minimum

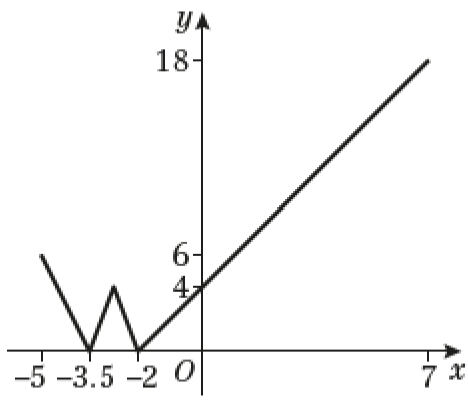
$\left(\frac{5}{2}, 5\ln\frac{5}{2} - \frac{13}{4}\right) \to \left(\frac{9}{2}, \frac{39}{4} - 15\ln\frac{5}{2}\right)$

Maximum

23 a The range of f(x) is $-2 \leqslant \mathrm{f}(x) \leqslant 18$

b ff(−3) = f(−2)
Using f(x) = 2x + 4
f(−2) = 2 × (−2) + 4 = 0

c



23 d Look at each section of $f(x)$ separately.

$-5 \leqslant x \leqslant -3$:

Gradient $= \frac{-2-6}{-3-(-5)} = -4$

$\therefore f(x) - (-2) = -4(x-(-3)) \Rightarrow f(x) = -4x - 14$

So in this region, $f(x) = 2$ when $x = -4$

$\therefore fg(x) = 2$ has a corresponding solution if

$g(x) = -4 \Rightarrow g(x) + 4 = x^2 - 7x + 14 = 0$

Discriminant $(-7)^2 - 4(1)(14) = -7 < 0$

So no solution

$-3 \leqslant x \leqslant 7$: Gradient $= \frac{18-(-2)}{7-(-3)} = 2$

$\therefore f(x) - (-2) = 2(x-(-3)) \Rightarrow f(x) = 2x + 4$

So in this region, $f(x) = 2$ when $x = -1$

$\therefore fg(x) = 2$ has a corresponding solution if

$g(x) = -1 \Rightarrow g(x) + 1 = x^2 - 7x + 11 = 0$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(11)}}{2(1)} = \frac{7 \pm \sqrt{5}}{2}$$

$\therefore x = \frac{7+\sqrt{5}}{2}$ or $x = \frac{7-\sqrt{5}}{2}$

24 a The range of p(x) is p(x) $\leqslant 10$

b p(x) is many-to-one, therefore the inverse is one-to-many, which is not a function.

c At first point of intersection:

$2(x + 4) + 10 = -4$

$2x + 18 = -4$

$x = -11$

At the other point of intersection:

$-2(x + 4) + 10 = -4$

$-2x + 2 = -4$

$x = 3$

$-11 < x < 3$

d For no solutions, p(x) > 10 at $x = -4$

So $-\frac{1}{2}x + k > 10$ at $x = -4$

$-\frac{1}{2}(-4) + k > 10$

$2 + k > 10$

$k > 8$

25 a Completing the square

$$\begin{aligned} 3x^2 - 12x + 20 &= 3(x^2 - 4x) + 20 \\ &= 3((x-2)^2 - 4) + 20 \\ &= 3(x-2)^2 - 12 + 20 \\ &= 3(x-2)^2 + 8 \end{aligned}$$

b $g(x) = \frac{1}{3x^2 - 12x + 20} = \frac{1}{3(x-2)^2 + 8}$

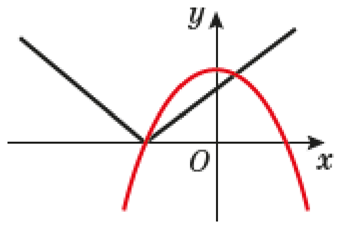
The maximum value of g(x) is $\frac{1}{8}$ (when $x = 2$)

As x approaches infinity, g(x) approaches 0

Therefore the range is $0 < g(x) \leqslant \frac{1}{8}$

Challenge

a



b $y = (a + x)(a - x)$

When $y = 0$, $x = -a$ or $x = a$

When $x = 0$, $y = a^2$

$(-a, 0)$, $(a, 0)$, $(0, a^2)$

c When $x = 4$, $y = a^2 - x^2$

$= a^2 - 16$

and $y = x + a$

$= 4 + a$

$a^2 - 16 = 4 + a$

$a^2 - a - 20 = 0$

$(a - 5)(a + 4) = 0$

As $a > 1$, $a = 5$