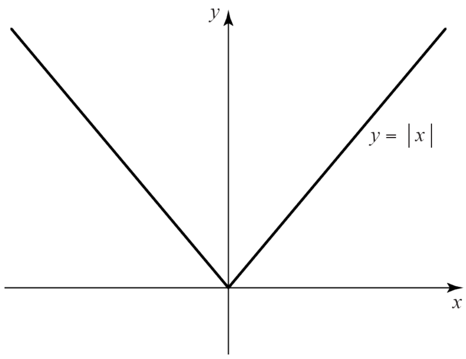
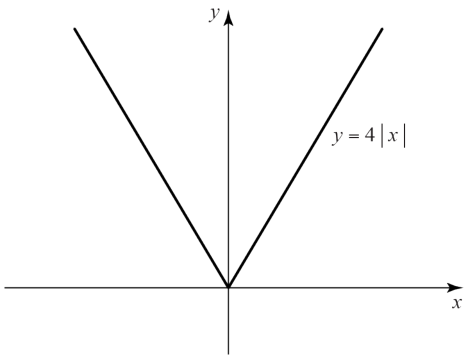
Functions and graphs 2G

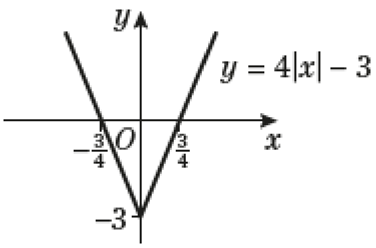
1 a Start with $y = |x|$



$y = 4|x|$ is a vertical stretch by scale factor 4

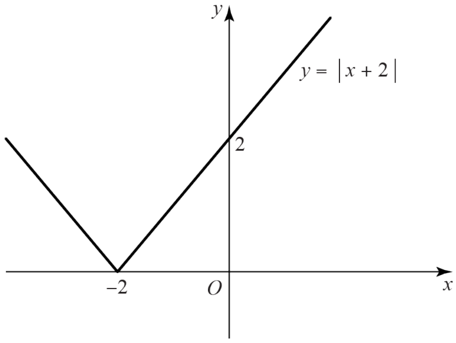


$y = 4|x| - 3$ is a horizontal translation by –3



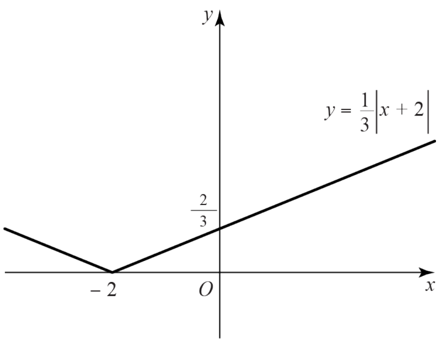
The range is $f(x) \geq -3$

b Start with $y = |x|$

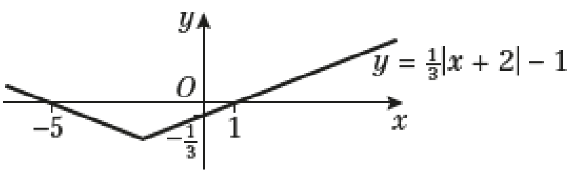


$y = |x+2|$ is a horizontal translation by –2

$y = \frac{1}{3}|x+2|$ is a vertical stretch by scale factor $\frac{1}{3}$

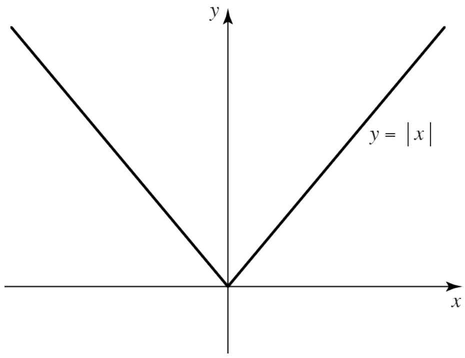


$y = \frac{1}{3}|x+2| - 1$ is a vertical translation by –1

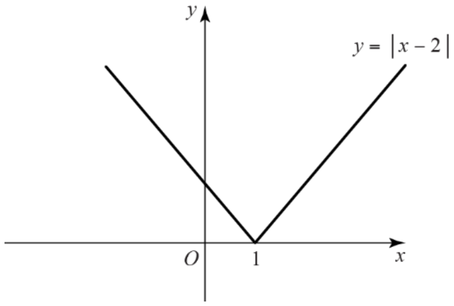


The range is $f(x) \geq -1$

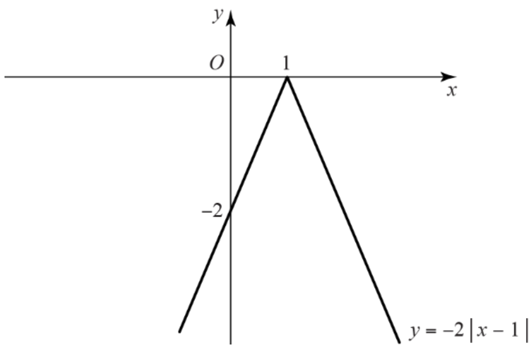
1 c Start with $y = |x|$



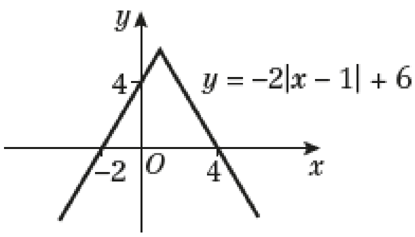
$y = |x-1|$ is a horizontal translation by +1



$y = -2|x-1|$ is a vertical stretch by scale factor –2

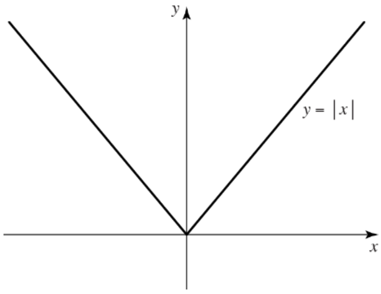


$y = -2|x-1|+6$ is a vertical translation by +6

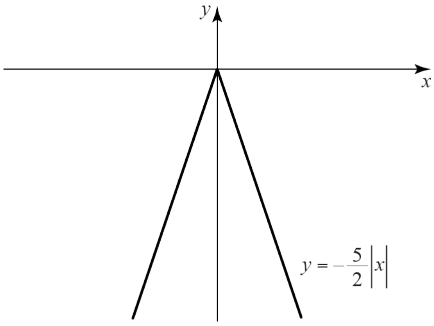


The range is $\mathrm{f}(x) \le 6$

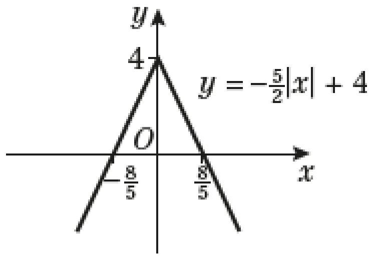
d Start with $y = |x|$



$y = -\frac{5}{2}|x|$ is a vertical stretch by scale factor $-\frac{5}{2}$



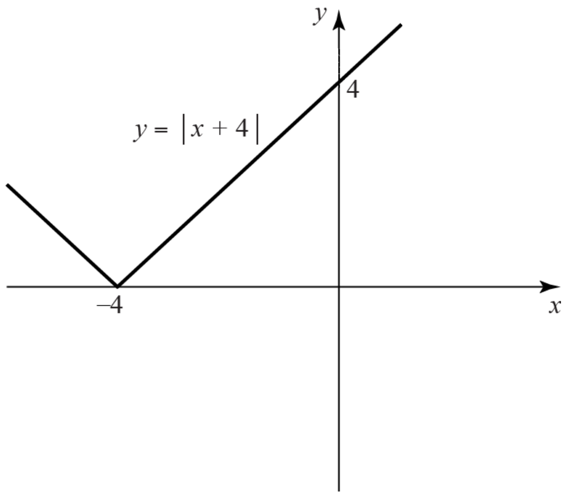
$y = -\frac{5}{2}|x| + 4$ is a horizontal translation by –3



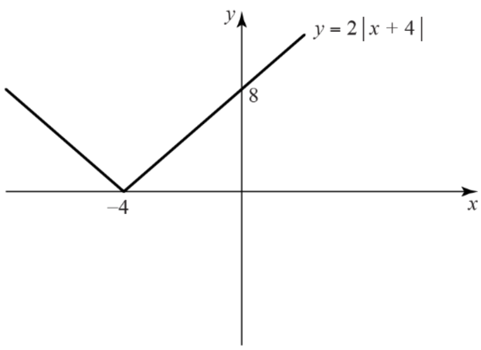
The range is $\mathrm{f}(x) \le 4$

2 a Start with $y = |x|$

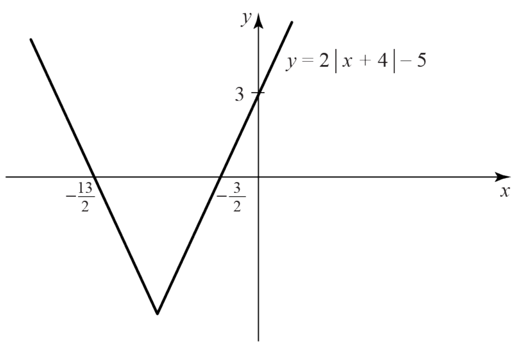
$y = |x+4|$ is a horizontal translation of –4



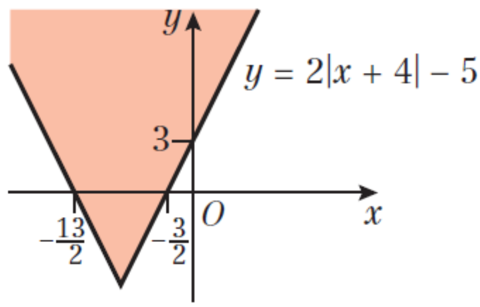
$y = 2|x+4|$ is a vertical stretch scale factor 2



$y = 2|x+4| - 5$ is a vertical translation of –5



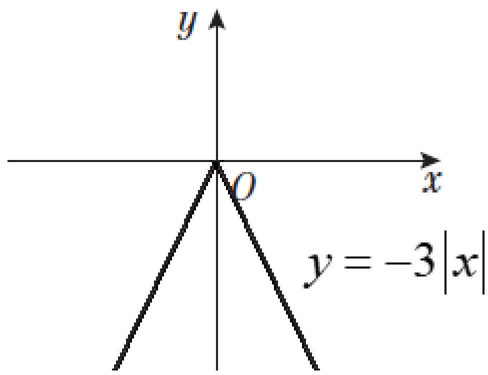
b The region where $y \geq \mathrm{p}(x)$ is the region which lies on and above the line $y = 2|x+4| - 5$



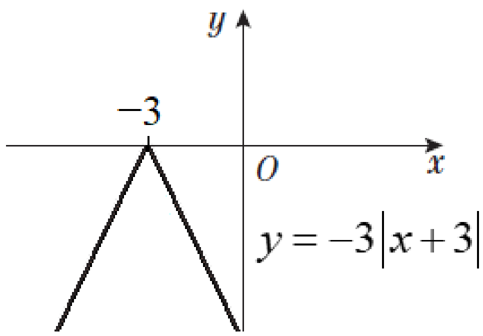
3 a $\mathrm{q}(x) = 6 - |3x+9| = -3|x+3| + 6$

Start with $y = |x|$

$y = -3|x|$ is a vertical stretch scale factor –3

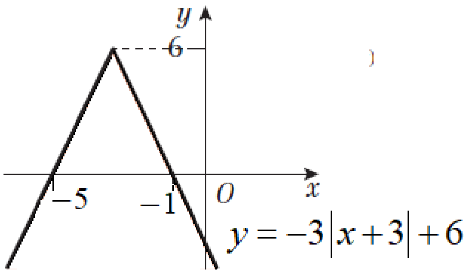


$y = -3|x+3|$ is a horizontal translation of −3

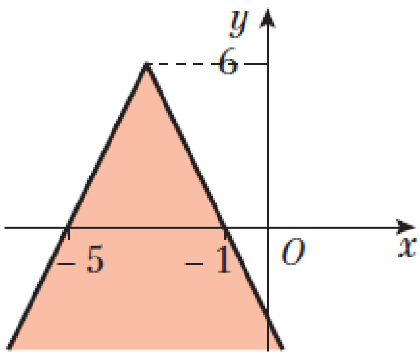


3 a (continued)

$y = -3|x+3| + 6$ is a vertical translation of +6

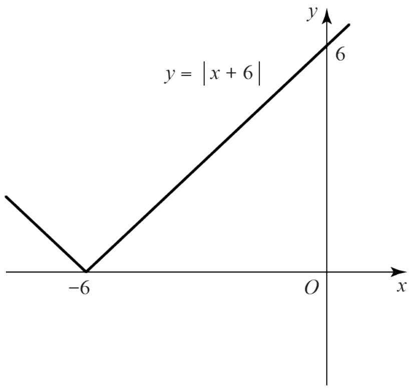


3 b The region where $y < q(x)$ is the region which lies below the line $y = -3|x+3| + 6$

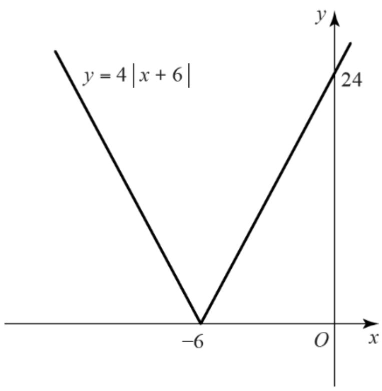


4 a Start with $y = |x|$

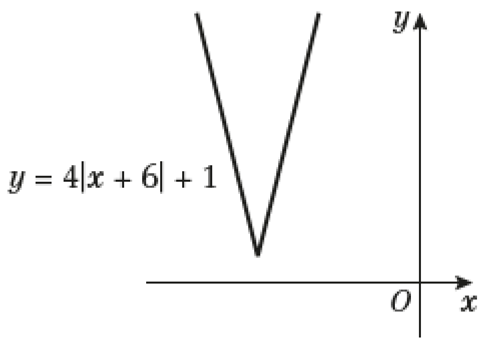
$y = |x+6|$ is a horizontal translation of –6



$y = 4|x+6|$ is a vertical stretch scale factor 4



$y = 4|x+6| + 1$ is a vertical translation of +1



4 b The range is $f(x) \geq 1$

c At one point of intersection:

$$-4(x+6)+1=-\frac{1}{2}x+1$$
$$-4x-23=-\frac{1}{2}x+1$$
$$-8x-46=-x+2$$
$$-48=7x$$
$$x=-\frac{48}{7}$$

At other point of intersection:

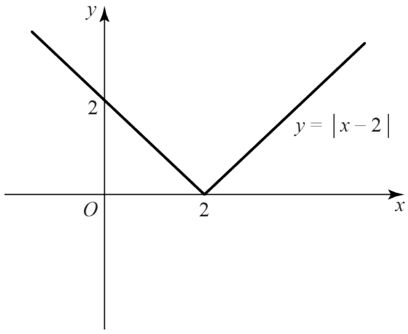
$$4(x+6)+1=-\frac{1}{2}x+1$$
$$4x+25=-\frac{1}{2}x+1$$
$$8x+50=-x+2$$
$$9x=-48$$
$$x=-\frac{16}{3}$$

So the solutions are

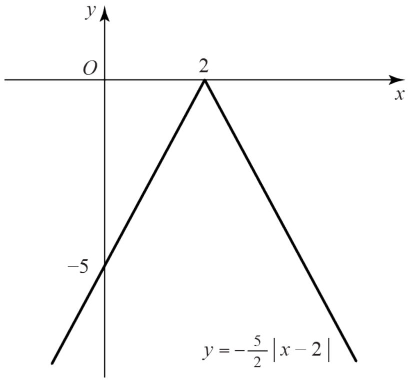
$x=-\frac{48}{7}$ and $x=-\frac{16}{3}$

5 a Start with $y=|x|$

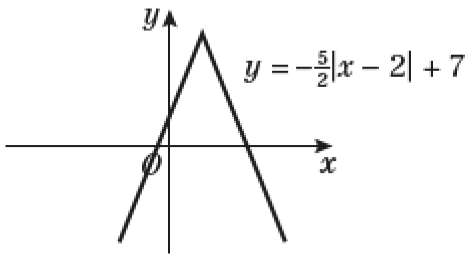
$y=|x-2|$ is a horizontal translation of +2



$y=-\frac{5}{2}|x-2|$ is a vertical stretch scale factor $-\frac{5}{2}$



$y=-\frac{5}{2}|x-2|+7$ is a vertical translation of +7



b The range is $g(x) \leq 7$

5 c At one point of intersection:

$$-\frac{5}{2}(x-2)+7=x+1$$
$$-\frac{5}{2}x+12=x+1$$
$$-5x+24=2x+2$$
$$22=7x$$
$$x=\frac{22}{7}$$

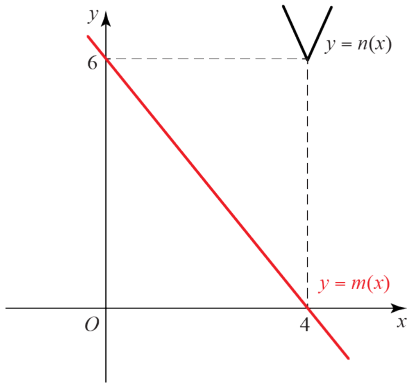
At other point of intersection:

$$\frac{5}{2}(x-2)+7=x+1$$
$$\frac{5}{2}x+2=x+1$$
$$5x+4=2x+2$$
$$3x=-2$$
$$x=-\frac{2}{3}$$

So the solutions are

$x=-\frac{2}{3}$ and $x=\frac{22}{7}$

6 For the equation $\mathrm{m}(x) = \mathrm{n}(x)$ to have no real roots, it must be the case that $y = \mathrm{m}(x)$ and $y = \mathrm{n}(x)$ do not intersect.



The least value of

$y=\mathrm{n}(x)=3|x-4|+6$ is

$y = 6$ when $x = 4$

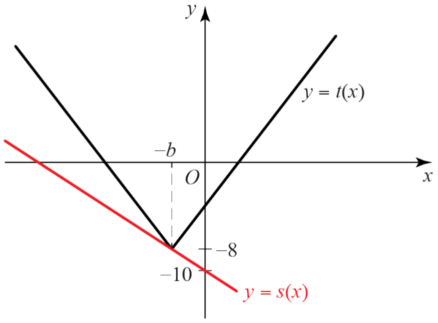
Hence, we need $\mathrm{m}(4) < 6$ to avoid intersection

So $-2(4) + k < 6$

$-8 + k < 6$

$k < 14$

7 For the equation s(x) = t(x) to have exactly one real root, it must be the case that y = s(x) and y = t(x) intersect at the minimum point of t(x).



The least value of

$y = \mathrm{t}(x) = 2|x+b| - 8$ is

$y = -8$ when $x = -b$

Hence, we need s(–b) = –8 to ensure one intersection

$\Rightarrow -8 = -10 - (-b)$

$b = 2$

8 a The range is $\mathrm{h}(x) \geq -7$

b h(x) is many-to-one, therefore $\mathrm{h}^{-1}(x)$ would be one-to-many, and so would not be a function.

c At one point of intersection:

$$-\frac{2}{3}(x-1) - 7 = -6$$
$$2x - 2 + 21 = 18$$
$$2x = -1$$
$$x = -\frac{1}{2}$$

At other point of intersection:

$$\frac{2}{3}(x-1) - 7 = -6$$
$$2x - 2 - 21 = -18$$
$$2x = 5$$
$$x = \frac{5}{2}$$

So the solutions are

$x = -\frac{1}{2}$ and $x = \frac{5}{2}$

h(x) < –6 between the two points of intersection, so the solution to the inequality $\mathrm{h}(x) < -6$ is

$$-\frac{1}{2} < x < \frac{5}{2}$$

d Since $\mathrm{h}(x) \geq -7$ and $\mathrm{h}(1) = -7$,

then for the equation $\mathrm{h}(x) = \frac{2}{3}x + k$

to have no solutions, we require

$$\frac{2}{3}(1) + k < -7$$
$$\Rightarrow k < -\frac{23}{3}$$

9 a We can write h as

$$h(x)=\begin{cases} a+2(x+3), & x\le -3 \\ a-2(x+3), & x\ge -3 \end{cases}$$

The line which has gradient –2 and passes through (0, 4) is $y=-2x+4$

So, for $x\ge -3$

$$\begin{aligned} -2(x+3)+a &= -2x+4 \\ -2x-6+a &= -2x+4 \\ a &= 10 \end{aligned}$$

b At P, h(x) = 10 (from part **a**)

$$\begin{aligned} \text{So}\quad 10 &= 10-2(x+3) \\ -2x-6 &= 0 \\ x &= -3 \end{aligned}$$

At Q, h(x) = 0

$$\begin{aligned} \text{So } 0 &= 10-2(x+3) \\ 4-2x &= 0 \\ x &= 2 \end{aligned}$$

$P(-3, 10)$ and $Q(2, 0)$

c $h(x)=\frac{1}{3}x+6$

At one point of intersection:

$$\begin{aligned} 10-2(x+3) &= \frac{1}{3}x+6 \\ 4-2x &= \frac{1}{3}x+6 \\ 12-6x &= x+18 \\ 7x &= -6 \\ x &= -\frac{6}{7} \end{aligned}$$

At other point of intersection:

$$\begin{aligned} 10+2(x+3) &= \frac{1}{3}x+6 \\ 16+2x &= \frac{1}{3}x+6 \\ 48+6x &= x+18 \\ 5x &= -30 \\ x &= -6 \end{aligned}$$

So the solutions are

$x=-6$ and $x=-\frac{6}{7}$

10 a The range of m(x) is $m(x)\le 7$

b $m(x)=\frac{3}{5}x+2$

At one point of intersection:

$$\begin{aligned} -4(x+3)+7 &= \frac{3}{5}x+2 \\ -4x-5 &= \frac{3}{5}x+2 \\ -20x-25 &= 3x+10 \\ -23x &= 35 \\ x &= -\frac{35}{23} \end{aligned}$$

At other point of intersection:

$$\begin{aligned} 4(x+3)+7 &= \frac{3}{5}x+2 \\ 4x+19 &= \frac{3}{5}x+2 \\ 20x+95 &= 3x+10 \\ 17x &= -85 \\ x &= -5 \end{aligned}$$

So the solutions are $x=-5$ and $x=-\frac{35}{23}$

c For two distinct roots, there are two points of intersection, so $m(x) < 7$. Therefore, $k < 7$.

Challenge

1 a At A:

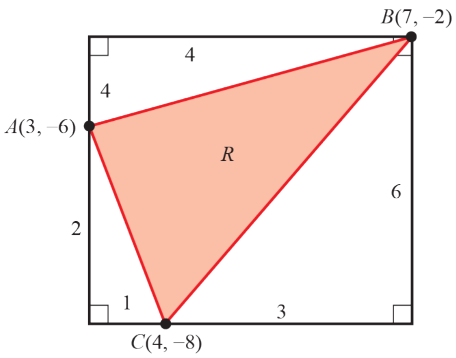
$-2(x-4)-8=x-9$
$-2x=x-9$
$-3x=-9$
$x=3$
$y=3-9=-6$

At B:

$2(x-4)-8=x-9$
$2x-16=x-9$
$x=7$
$y=7-9=-2$

$A(3, -6)$ and $B(7, -2)$

b Taking the shaded triangle R and enclosing it in a rectangle looks like:



$R=(4\times 6)-\left(\frac{1}{2}\times 4\times 4\right)-\left(\frac{1}{2}\times 6\times 3\right)-\left(\frac{1}{2}\times 2\times 1\right)$

$R=24-8-9-1$

$R=6$ units^2

2 At the first point of intersection:

$x-3+10=-2(x-3)+2$
$x+7=-2x+8$
$3x=1$
$x=\frac{1}{3}$

At the other point of intersection:

$-(x-3)+10=2(x-3)+2$
$-x+13=2x-4$
$-3x=-17$
$x=\frac{17}{3}$

Maximum point of $f(x)$ is
$f(x)=10$ when $x=3$, so at $(3, 10)$
Minimum point of $g(x)$ is
$g(x)=2$ when $x=3$, so at $(3, 2)$

Area of a kite $=\frac{1}{2}\times$ width $\times$ height

$=\frac{1}{2}\times\left(\frac{17}{3}-\frac{1}{3}\right)\times(10-2)$

$=\frac{1}{2}\times\frac{16}{3}\times 8$

$=\frac{64}{3}$ units^2