Algebraic methods 1A

1 B At least one multiple of three is odd.

2 **a** At least one rich person is not happy.

b There is at least one prime number between 10 million and 11 million.

c If p and q are prime numbers there exists a number of the form $(pq + 1)$ that is not prime.

d There is a number of the form $2^n - 1$ that is either not prime or not a multiple of 3.

e None of the above statements is true.

3 **a** There exists a number n such that n^2 is odd but n is even.

b n is even so write $n = 2k$
$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
So n^2 is even.
This contradicts the assumption that n^2 is odd.
Therefore, if n^2 is odd then n must be odd.

4 **a** Assumption: there is a greatest even integer, $2n$.
$2(n + 1)$ is also an integer and
$2(n + 1) > 2n$
$2n + 2 = \text{even} + \text{even} = \text{even}$
So there exists an even integer greater that $2n$.
This contradicts the assumption that the greatest even integer is $2n$.
Therefore there is no greatest even integer.

b Assumption: there exists a number n such that n^3 is even but n is odd.
n is odd so write $n = 2k + 1$
$n^3 = (2k + 1)^3$
$= 8k^3 + 12k^2 + 6k + 1$
$= 2(4k^3 + 6k^2 + 3k) + 1$
So n^3 is odd.
This contradicts the assumption that n^3 is even.
Therefore, if n^3 is even then n must be even.

c Assumption: if pq is even then neither p nor q is even.
p is odd, $p = 2k + 1$
q is odd, $q = 2m + 1$
$pq = (2k + 1)(2m + 1)$
$= 2km + 2k + 2m + 1$
$= 2(km + k + m) + 1$
So pq is odd.
This contradicts the assumption that pq is even.
Therefore, if pq is even then at least one of p and q is even.

4 d Assumption: if $p + q$ is odd than neither p nor q is odd.
p is even, $p = 2k$
q is even, $q = 2m$
$p + q = 2k + 2m = 2(k + m)$
So $p + q$ is even.
This contradicts the assumption that $p + q$ is odd.
Therefore, if $p + q$ is odd then at least one of p and q is odd.

5 a Assumption: if ab is an irrational number then neither a nor b is irrational.
a is rational, $a = \frac{c}{d}$ where c and d are integers.
b is rational, $b = \frac{e}{f}$ where e and f are integers.
$ab = \frac{ce}{df}$, ce is an integer, df is an integer.
Therefore ab is a rational number.
This contradicts assumption that ab is irrational.
Therefore, if ab is an irrational number then at least one of a and b is an irrational number.

b Assumption: if $a + b$ is an irrational number then neither a nor b is irrational.
a is rational, $a = \frac{c}{d}$ where c and d are integers.
b is rational, $b = \frac{e}{f}$ where e and f are integers.
$a + b = \frac{cf + de}{df}$, cf, de and df are integers.
So $a + b$ is rational. This contradicts the assumption that $a + b$ is irrational.
Therefore if $a + b$ is irrational then at least one of a and b is irrational.

c Many possible answers
e.g. $a = 2 - \sqrt{2}$, $b = \sqrt{2}$.

6 Assumption: there exist integers a and b such that $21a + 14b = 1$.
Since 21 and 14 are multiples of 7, divide both sides by 7.
So now $3a + 2b = \frac{1}{7}$
$3a$ is also an integer. $2b$ is also an integer.
The sum of two integers will always be an integer, so $3a + 2b$ is an integer.
This contradicts the statement that
$3a + 2b = \frac{1}{7}$
Therefore there exist no integers a and b for which $21a + 14b = 1$.

7 **a** Assumption: There exists a number n such that n^2 is a multiple of 3, but n is not a multiple of 3.
All multiples of 3 can be written in the form $n = 3k$ where k is an integer, therefore $3k + 1$ and $3k + 2$ are not multiples of 3.
Let $n = 3k + 1$
$n^2 = (3k + 1)^2$
$= 9k^2 + 6k + 1$
$= 3(3k^2 + 2k) + 1$
In this case n^2 is not a multiple of 3.
Let $n = 3k + 2$
$n^2 = (3k + 2)^2$
$= 9k^2 + 12k + 4$
$= 3(3k^2 + 4k + 1) + 1$
In this case n^2 is also not a multiple of 3.
This contradicts the assumption that n^2 is a multiple of 3.
Therefore if n^2 is a multiple of 3, n is a multiple of 3.

b Assumption: $\sqrt{3}$ is a rational number.
Then $\sqrt{3} = \frac{a}{b}$ for some integers a and b.
Further assume that this fraction is in its simplest terms: there are no common factors between a and b.
So $3 = \frac{a^2}{b^2}$ or $a^2 = 3b^2$
Therefore a^2 must be a multiple of 3.
We know from part **a** that this means a must also be a multiple of 3.
Write $a = 3c$, which means $a^2 = (3c)^2 = 9c^2$.
Now $9c^2 = 3b^2$, or $3c^2 = b^2$.
Therefore b^2 must be a multiple of 3, which means b is also a multiple of 3.
If a and b are both multiples of 3, this contradicts the statement that there are no common factors between a and b.
Therefore, $\sqrt{3}$ is an irrational number.

8 Assumption: there is an integer solution to the equation $x^2 – y^2 = 2$.
Remember that $x^2 – y^2 = (x – y)(x + y) = 2$.
To make a product of 2 using integers, the possible pairs are: (2, 1), (1, 2), (–2, –1) and (–1, –2).
Consider each possibility in turn:
$x – y = 2$ and $x + y = 1 \Rightarrow x = \frac{3}{2}, y = -\frac{1}{2}$
$x – y = 1$ and $x + y = 2 \Rightarrow x = \frac{3}{2}, y = \frac{1}{2}$
$x – y = –2$ and $x + y = –1 \Rightarrow x = -\frac{3}{2}, y = \frac{1}{2}$
$x – y = –1$ and $x + y = –2 \Rightarrow x = -\frac{3}{2}, y = -\frac{1}{2}$
This contradicts the statement that there is an integer solution to the equation $x^2 – y^2 = 2$.
Therefore the original statement must be true:

8 (continued)
There are no integer solutions to the equation $x^2 - y^2 = 2$.

9 Assumption: $\sqrt[3]{2}$ is a rational number
Then $\sqrt[3]{2} = \frac{a}{b}$ for some integers a and b.
Further assume that this fraction is in its simplest terms:
there are no common factors between a and b.
This means that if a^3 is even, a must also be even.
If a is even, $a = 2n$.
So $a^3 = 2b^3$ becomes $(2n)^3 = 2b^3$ which means $8n^3 = 2b^3$ or $4n^3 = b^3$ or $2(2n^3) = b^3$
This means that b^3 must be even, so b is also even.
If a and b are both even, they will have a common factor of 2.
This contradicts the statement that a and b have no common factors.
We can conclude the original statement is true: $\sqrt[3]{2}$ is an irrational number.

10 a The number $\frac{a-b}{b}$ could be negative.
e.g. If $n = \frac{1}{2}$, $n - 1$ is non-positive.

b Assumption: There is a least positive rational number, n.
$n = \frac{a}{b}$ where a and b are integers.
Let $m = \frac{a}{2b}$. Since a and b are integers, m is rational and $m < n$.
This contradicts the statement that n is the least positive rational number.
Therefore, there is no least positive rational number.