Review exercise 3

1 The two vectors are parallel
so $9\mathbf{i} + q\mathbf{j} = \lambda(2\mathbf{i} - \mathbf{j})$

Equating coefficients:
$9 = 2\lambda$
$\lambda = 4.5$
$q = -\lambda$
$= -4.5$

2 $|5\mathbf{i} - k\mathbf{j}| = |2k\mathbf{i} + 2\mathbf{j}|$
$\sqrt{5^2 + k^2} = \sqrt{(2k)^2 + 2^2}$
$25 + k^2 = 4k^2 + 4$
$3k^2 = 21$
$k^2 = 7$
$k = \pm\sqrt{7}$
The positive value of k is $\sqrt{7}$.

3 a $\overrightarrow{CX} = \begin{pmatrix} 1-9 \\ -3-6 \end{pmatrix} = \begin{pmatrix} -8 \\ -9 \end{pmatrix}$
$|\overrightarrow{CX}| = \sqrt{8^2 + 9^2} = \sqrt{145}$
$\overrightarrow{CY} = \begin{pmatrix} 1-13 \\ -3+2 \end{pmatrix} = \begin{pmatrix} -12 \\ -1 \end{pmatrix}$
$|\overrightarrow{CY}| = \sqrt{12^2 + 1^2} = \sqrt{145}$
$\overrightarrow{CZ} = \begin{pmatrix} 1-0 \\ -3+15 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$
$|\overrightarrow{CZ}| = \sqrt{1^2 + 12^2} = \sqrt{145}$
Therefore, $|\overrightarrow{CX}| = |\overrightarrow{CY}| = |\overrightarrow{CZ}|$

b Centre of the circle is point $C(1, -3)$.
Radius of the circle is $\sqrt{145}$.
Equation of the circle is
$(x - 1)^2 + (y + 3)^2 = 145$

4 a $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$
$= -(9\mathbf{i} + 2\mathbf{j}) + (7\mathbf{i} - 6\mathbf{j})$
$= -2\mathbf{i} - 8\mathbf{j}$

b For triangle ABC to be isosceles two of the sides must be equal.
$AB = \sqrt{9^2 + 2^2} = \sqrt{85}$
$BC = \sqrt{2^2 + 8^2} = \sqrt{68}$
$AC = \sqrt{7^2 + 6^2} = \sqrt{85}$
$AB = AC$, therefore triangle ABC is isosceles

4 c Using the cosine rule
$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$\cos B = \dfrac{\sqrt{68}^2 + \sqrt{85}^2 - \sqrt{85}^2}{2\sqrt{68}\sqrt{85}}$
$\cos B = \dfrac{68 + 85 - 85}{2\sqrt{5780}}$
$\cos B = \dfrac{68}{68\sqrt{5}}$
$\cos B = \dfrac{1}{\sqrt{5}}$
So $\cos \angle ABC = \dfrac{1}{\sqrt{5}}$

5 $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 8 \\ 23 \end{pmatrix} + \begin{pmatrix} -15 \\ x \end{pmatrix} = \begin{pmatrix} -7 \\ 23 + x \end{pmatrix}$

or : $-7\mathbf{i} + (23 + x)\mathbf{j}$

$\mathbf{b} - \mathbf{c} = \begin{pmatrix} -15 \\ x \end{pmatrix} - \begin{pmatrix} -13 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ x - 2 \end{pmatrix}$

or : $-2\mathbf{i} + (x - 2)\mathbf{j}$

As $\mathbf{a} + \mathbf{b}$ is parallel to $\mathbf{b} - \mathbf{c}$
$-7\mathbf{i} + (23 + x)\mathbf{j} = \lambda(-2\mathbf{i} + (x - 2)\mathbf{j})$

Equating coefficients and solving simultaneously
$-7 = -2\lambda$ and $23 + x = \lambda(x - 2)$
$\lambda = 3.5$
$23 + x = 3.5(x - 2)$
$23 + x = 3.5x - 7$
$2.5x = 30$
$x = 12$

6 a $\mathbf{R} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{i} + \mathbf{j}$
$= 3\mathbf{i} - 4\mathbf{j}$
$|\mathbf{R}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ N

b $R_{new} = 3\mathbf{i} - 4\mathbf{j} + k\mathbf{j}$
$\tan 45° = 1$, so the coefficients of $\mathbf{i}$ and $\mathbf{j}$ are equal.
So $-4\mathbf{j} + k\mathbf{j} = 3\mathbf{i}$
So $k = 7$

7 $\sin 60° = \dfrac{x}{100}$

$x = 100 \sin 60°$

$= 50\sqrt{3}$

Using Pythagoras' theorem:

$y = \sqrt{100^2 - (50\sqrt{3})^2} = \sqrt{2500} = 50$

or using $\cos 60° = \dfrac{y}{100}$ so $y = 50$

$m = 50\sqrt{3} + 30$ and $n = 50$

8 a Call the finish line F:

$\overrightarrow{AF} = -65\mathbf{i} + 180\mathbf{j} - 10\mathbf{i} = -75\mathbf{i} + 180\mathbf{j}$

$AF = \sqrt{75^2 + 180^2} = \sqrt{38025} = 195$

$\overrightarrow{BF} = 100\mathbf{i} + 120\mathbf{j} - 10\mathbf{i} = 90\mathbf{i} + 120\mathbf{j}$

$BF = \sqrt{90^2 + 120^2} = \sqrt{22500} = 150$

150 < 195, so boat B is closer to the finish line.

b Speed of boat $A = \sqrt{2.5^2 + 6^2}$

$= \sqrt{42.25}$

$= 6.5$ m/s

Speed of boat $B = \sqrt{3^2 + 4^2}$

$= \sqrt{25}$

$= 5$ m/s

$\text{Time} = \dfrac{\text{Distance}}{\text{Speed}}$

Time taken for boat A to reach the finish line $= \dfrac{195}{6.5} = 30$ s

Time taken for boat B to reach the finish line $= \dfrac{150}{5} = 30$ s

Both boats reach the finish line at the same time.

9 $f(x) = 5x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0}\frac{5(x+h)^2 - 5x^2}{h}$$
$$= \lim_{h\to 0}\frac{5x^2 + 10xh + 5h^2 - 5x^2}{h}$$
$$= \lim_{h\to 0}\frac{10xh + 5h^2}{h}$$
$$= \lim_{h\to 0}\frac{h(10x + 5h)}{h}$$
$$= \lim_{h\to 0}(10x + 5h)$$

As $h \to 0$, $10x + 5h \to 10x$, so $f'(x) = 10x$

10 $y = 4x^3 - 1 + 2x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \left(4\times 3x^2\right) + \left(2\times\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$\frac{dy}{dx} = 12x^2 + x^{-\frac{1}{2}}$$

Or:

$$\frac{dy}{dx} = 12x^2 + \frac{1}{x^{\frac{1}{2}}}$$

Or:

$$\frac{dy}{dx} = 12x^2 + \frac{1}{\sqrt{x}}$$

11 a $y = 4x + 3x^{\frac{3}{2}} - 2x^2$

$$\frac{dy}{dx} = \left(4\times 1x^0\right) + \left(3\times\tfrac{3}{2}x^{\frac{1}{2}}\right) - \left(2\times 2x^1\right)$$
$$\frac{dy}{dx} = 4 + \tfrac{9}{2}x^{\frac{1}{2}} - 4x$$

b For $x = 4$,

$$y = (4\times 4) + \left(3\times 4^{\frac{3}{2}}\right) - \left(2\times 4^2\right)$$
$$= 16 + (3\times 8) - 32$$
$$= 16 + 24 - 32$$
$$= 8$$

So P (4, 8) lies on C.

11 c For $x = 4$,

$$\frac{dy}{dx} = 4 + \left(\tfrac{9}{2} \times 4^{\frac{1}{2}}\right) - (4 \times 4)$$
$$= 4 + \left(\tfrac{9}{2} \times 2\right) - 16$$
$$= 4 + 9 - 16$$
$$= -3$$

This is the gradient of the tangent.

The gradient of the normal at P is $\frac{1}{3}$.

The normal is perpendicular to the tangent, so the gradient is $-\frac{1}{m}$.

Equation of the normal:

$$y - 8 = \frac{1}{3}(x - 4)$$
$$y - 8 = \frac{x}{3} - \frac{4}{3}$$
$$3y - 24 = x - 4$$
$$3y = x + 20$$
$$y = \frac{1}{3}x + \frac{20}{3}$$

d $y = 0$:

$0 = x + 20$

$x = -20$

Q is the point $(-20, 0)$.

$$PQ = \sqrt{(4 - -20)^2 + (8 - 0)^2}$$
$$= \sqrt{24^2 + 8^2}$$
$$= \sqrt{576 + 64}$$
$$= \sqrt{640}$$
$$= \sqrt{64} \times \sqrt{10}$$
$$= 8\sqrt{10}$$

12 a $y = 4x^2 + \frac{5 - x}{x}$

$$= 4x^2 + 5x^{-1} - 1$$

$$\frac{dy}{dx} = \left(4 \times 2x^1\right) + \left(5x \times -1x^{-2}\right)$$
$$\frac{dy}{dx} = 8x - 5x^{-2}$$

At P, $x = 1$, so

$$\frac{dy}{dx} = (8 \times 1) - \left(5 \times 1^{-2}\right)$$
$$= 8 - 5 = 3$$

12 b At $x = 1$, $\frac{dy}{dx} = 3$

The value of $\frac{dy}{dx}$ is the gradient of the tangent.

At $x = 1$, $y = \left(4 \times 1^2\right) + \frac{5 - 1}{1}$

$$y = 4 + 4 = 8$$

Equation of the tangent:

$y - 8 = 3(x - 1)$

$y = 3x + 5$

c $y = 0 : 0 = 3x + 5$

$3 = -5$

$x = -\frac{5}{3}$

So $k = -\frac{5}{3}$

13 a $f(x) = \frac{(2x + 1)(x + 4)}{\sqrt{x}}$

$$= \frac{2x^2 + 9x + 4}{\sqrt{x}}$$
$$= 2x^{\frac{3}{2}} + 9x^{\frac{1}{2}} + 4x^{\frac{-1}{2}}$$

$P = 2,\ Q = 9,\ R = 4$

b $f'(x) = \left(2 \times \tfrac{3}{2} x^{\frac{1}{2}}\right) + \left(9 \times \tfrac{1}{2} x^{-\frac{1}{2}}\right) + \left(4 \times -\tfrac{1}{2} x^{-\frac{3}{2}}\right)$

$$f'(x) = 3x^{\frac{1}{2}} + \tfrac{9}{2} x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}}$$

c At $x = 1$,

$$f'(1) = \left(3 \times 1^{\frac{1}{2}}\right) + \left(\tfrac{9}{2} \times 1^{-\frac{1}{2}}\right) - \left(2 \times 1^{-\frac{3}{2}}\right)$$
$$= 3 + \tfrac{9}{2} - 2$$
$$= \tfrac{11}{2}$$

The line $2y = 11x + 3$ is

$$y = \tfrac{11}{2} x + \tfrac{3}{2}$$

$\therefore$ The gradient is $\frac{11}{2}$.

The tangent to the curve where $x = 1$ is parallel to this line, since the gradients are equal.

14 $f(x) = x^3 - 12x^2 + 48x$
$f'(x) = 3x^2 - 24x + 48$
$= 3(x - 4)^2$
$(x - 4)^2 > 0$ for all real values of x

So $3x^2 - 24x + 48 > 0$ for all real values of x.

So $f(x)$ is increasing for all real values of x.

15 a $y = x + \frac{2}{x} - 3$

When $y = 0$, $x + \frac{2}{x} - 3 = 0$
$x^2 + 2 - 3x = 0$
$x^2 - 3x + 2 = 0$
$(x - 1)(x - 2) = 0$
$x = 1$ or $x = 2$

$A(1, 0)$ and $B(2, 0)$

b $y = x + 2x^{-1} - 3$

$\frac{dy}{dx} = 1 - 2x^{-2}$

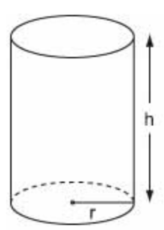
$= 1 - \frac{2}{x^2}$

Let $\frac{dy}{dx} = 0$ to find the minimum

$x^2 = 2$
$x = \pm\sqrt{2}$
x is positive, so $x = \sqrt{2}$.
When $x = \sqrt{2}$,
$y = \sqrt{2} + \frac{2}{\sqrt{2}} - 3$
$= \sqrt{2} + \frac{2\sqrt{2}}{2} - 3$
$= 2\sqrt{2} - 3$

C has coordinates $(\sqrt{2}, 2\sqrt{2} - 3)$

16



16 Draw a diagram. Let h be the height of the cylinder.

a Surface area, $S = 2\pi rh + 2\pi r^2$
Volume $= \pi r^2 h = 128\pi$
$h = \frac{128\pi}{\pi r^2}$
$= \frac{128}{r^2}$

so $S = 2\pi r \times \frac{128}{r^2} + 2\pi r^2$
$= \frac{256\pi}{r} + 2\pi r^2$ (as required)

b $\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$

$4\pi r - \frac{256\pi}{r^2} = 0$

$4\pi r = \frac{256\pi}{r^2}$

$r^3 = 64$

$r = 4\,\text{cm}$

When $r = 4$,

$S = \frac{256\pi}{(4)} + 2\pi(4)^2$

$= 64\pi + 32\pi$

$= 96\pi\,\text{cm}^2$

17 a $y = 3x^2 + 4\sqrt{x}$

$= 3x^2 + 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = (3 \times 2x^1) + \left(4 \times \frac{1}{2}x^{-\frac{1}{2}}\right)$

$\frac{dy}{dx} = 6x + 2x^{-\frac{1}{2}}$

Or:

$\frac{dy}{dx} = 6x + \frac{2}{x^{\frac{1}{2}}} = 6x + \frac{2}{\sqrt{x}}$

b $\frac{dy}{dx} = 6x + 2x^{\frac{-1}{2}}$

$\frac{d^2y}{dx^2} = 6 + \left(2 \times -\frac{1}{2}x^{-\frac{3}{2}}\right)$

$= 6 - x^{\frac{-3}{2}}$

17 b Or:

$$\frac{d^2y}{dx^2}=6-\frac{1}{x^{\frac{3}{2}}}$$

Or :

$$\frac{d^2y}{dx^2}=6-\frac{1}{x\sqrt{x}}$$

c $$\int\left(3x^2+4x^{\frac{1}{2}}\right)dx=\frac{3x^3}{3}+\frac{4x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+C$$

$$=x^3+4\left(\tfrac{2}{3}\right)x^{\frac{3}{2}}+C$$

$$=x^3+\tfrac{8}{3}x^{\frac{3}{2}}+C$$

$$\left(\text{Or}: x^3+\tfrac{8}{3}x\sqrt{x}+C\right)$$

18 a $f'(x)=6x^2-10x-12$

$$f(x)=\frac{6x^3}{3}-\frac{10x^2}{2}-12x+C$$

When $x = 5$, $y = 65$, so:

$$65=\frac{6\times125}{3}-\frac{10\times25}{2}-60+C$$

$$65=250-125-60+C$$

$$C=65+125+60-250$$

$$C=0$$

$$f(x)=2x^3-5x^2-12x$$

b $f(x)=x\left(2x^2-5x-12\right)$

$f(x)=x(2x+3)(x-4)$

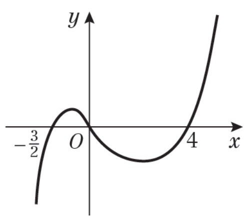
c Curve meets x-axis where $y=0$

$x(2x+3)(x-4)=0$

$x=0, x=-\frac{3}{2}, x=4$

When $x\to\infty, y\to\infty$

When $x\to-\infty, y\to-\infty$



Crosses x-axis at $\left(-\frac{3}{2},0\right)$, $(0,0)$ and $(4,0)$.

19 $$\int_1^8 x^{\frac{1}{3}}-x^{\frac{-1}{3}}\,dx$$

$$=\left[\tfrac{3}{4}x^{\frac{4}{3}}-\tfrac{3}{2}x^{\frac{2}{3}}\right]_1^8$$

$$=\left(\tfrac{3}{4}(8)^{\frac{4}{3}}-\tfrac{3}{2}(8)^{\frac{2}{3}}\right)-\left(\tfrac{3}{4}(1)^{\frac{4}{3}}-\tfrac{3}{2}(1)^{\frac{2}{3}}\right)$$

$$=\left(\tfrac{3}{4}(16)-\tfrac{3}{2}(4)\right)-\left(\tfrac{3}{4}(1)-\tfrac{3}{2}(1)\right)$$

$$=\frac{27}{4}$$

$$=6\frac{3}{4}$$

20 $$\int_0^6(x^2-kx)\,dx$$

$$=\left[\frac{x^3}{3}-\frac{kx^2}{2}\right]_0^6$$

$$=\left(\frac{6^3}{3}-\frac{k(6)^2}{2}\right)-\left(\frac{0^3}{3}-\frac{k(0)^2}{2}\right)$$

$$=72-18k$$

Given that $\int_0^6(x^2-kx)\,dx=0$

$$72-18k=0$$

$$k=4$$

21 a $-x^4+3x^2+4=0$

$(-x^2+4)(x^2+1)=0$

$(2-x)(2+x)(x^2+1)=0$

$x^2+1=0$ has no real solutions.

So there are two solutions $x=-2$ or $x=2$.

$A(-2, 0)$ and $B(2, 0)$

b $$R=\int_{-2}^{2}(-x^4+3x^2+4)\,dx$$

$$=\left[-\frac{x^5}{5}+\frac{3x^3}{3}+4x\right]_{-2}^{2}$$

$$=\left[-\frac{x^5}{5}+x^3+4x\right]_{-2}^{2}$$

$$=\left(-\frac{2^5}{5}+2^3+4(2)\right)-\left(-\frac{(-2)^5}{5}+(-2)^3+4(-2)\right)$$

$$=\left(-\frac{32}{5}+8+8\right)-\left(\frac{32}{5}-8-8\right)$$

$$=19.2\text{ units}^2$$

22 Area $= \int_1^4 (x-1)(x-4)\ dx$

$= \int_1^4 x^2 - 5x + 4\ dx$

$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_1^4$

$= \left(\frac{(4)^3}{3} - \frac{5(4)^2}{2} + 4(4)\right)$

$= -\left(\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 4(1)\right)$

$= -4\frac{1}{2}$

$\therefore$ Area $= 4\frac{1}{2}$ units^2 (area cannot be a negative value)

23 a Solving simultaneously

$5 - x^2 = 3 - x$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

when $x = 2$, $y = 1$

when $x = -1$, $y = 4$

$P(-1, 4)$ and $Q(2, 1)$

b Shaded area =

area under the curve between P and Q and the x-axis − area of trapezium

Area $= \int_{-1}^{2} (5 - x^2)\ dx - \frac{1}{2} \times 3(1 + 4)$

$= \left[5x - \frac{x^3}{3}\right]_{-1}^{2} - \frac{15}{2}$

$= \left(5(2) - \frac{2^3}{3}\right) - \left(5(-1) - \frac{(-1)^3}{3}\right) - \frac{15}{2}$

$= \left(10 - \frac{8}{3}\right) - \left(-5 + \frac{1}{3}\right) - \frac{15}{2}$

$= 4.5$ units^2

24 a $k = -1$

At point A, $x = 0$

$f(x) = 3e^0 - 1$

$= 2$

$A(0, 2)$ The y-coordinate of A is 2.

24 b At point B, $y = 0$

$3e^{-x} - 1 = 0$

$3e^{-x} = 1$

$e^{-x} = \frac{1}{3}$

$\ln(e^{-x}) = \ln\frac{1}{3}$

$-x = \ln\frac{1}{3}$

$x = -\ln\frac{1}{3}$

$= \ln\left(\frac{1}{3}\right)^{-1}$

$= \ln 3$ (which is the x-coordinate of B)

25 $T = 400e^{-0.05t} + 25, \quad t \geq 0$

a let $t = 0$

$T = 400 \times e^0 + 25 = 425°C$

b let $T = 300$

$300 = 400e^{-0.05t} + 25$

$300 - 25 = 400e^{-0.05t}$

$275 = 400e^{-0.05t}$

$\frac{275}{400} = e^{-0.05t}$

Take ln of both sides:

$\ln\left(\frac{275}{400}\right) = -0.05t$

$\frac{-1}{0.05}\ln\left(\frac{275}{400}\right) = t$

$t = 7.49$ minutes

c $T = 400e^{-0.05t} + 25$

$\frac{dT}{dt} = 400e^{-0.05t} \times -0.05$

$= -20e^{-0.05t}$

let $t = 50$

$\frac{dT}{dt} = -20e^{-0.05t \times 50}$

$= -20e^{-2.5}$

$= 1.64$

The rate the temperature is decreasing is 1.64°C/min

d $T = 400e^{-0.05t} + 25, \quad t \geq 0$

$e^{-0.05t}$ tends to 0, so effectively the minimum value of T is 25°C. Therefore, 20°C is not possible.

25 e In the given model, the temperature after a long period of time is 25 °C.

Replace 25 with 15 to give:

$T = 410e^{-0.05t} + 15,\ t \geq 0$

26 a $5^x = 0.75$

$x \log 5 = \log 0.75$

$x = \dfrac{\log 0.75}{\log 5}$

$x = -0.179$

b $2\log_5 x - \log_5 3x = 1$

$\log_5 x^2 - \log_5 3x = 1$

$\log_5\left(\dfrac{x^2}{3x}\right) = 1$

$5^1 = \dfrac{x^2}{3x} = \dfrac{x}{3}$

$x = 15$

27 a $3^{2x-1} = 10$

$(2x-1)\log 3 = \log 10$

$2x - 1 = \dfrac{\log 10}{\log 3}$

$2x = \dfrac{1}{\log 3} + 1 \quad (\log 10 = 1)$

$x = \dfrac{1}{2}\left(\dfrac{1}{\log 3} + 1\right)$

$= 1.55$

b $\log_2 x + \log_2(9-2x) = 2$

$\log_2 x(9-2x) = 2$

$2^2 = x(9-2x)$

$4 = 9x - 2x^2$

$2x^2 - 9x + 4 = 0$

$(2x-1)(x-4) = 0$

$x = \frac{1}{2}$ or $x = 4$

28 a $\log_p 12 - \left(\frac{1}{2}\log_p 9 + \frac{1}{3}\log_p 8\right)$

$= \log_p 12 - \left(\log_p 9^{\frac{1}{2}} + \log_p 8^{\frac{1}{3}}\right)$

$= \log_p 12 - \left(\log_p 3 + \log_p 2\right)$

$= \log_p 12 - \left(\log_p (3 \times 2)\right)$

$= \log_p 12 - \log_p 6$

28 a $= \log_p\left(\frac{12}{6}\right)$

$= \log_p 2$

b $\log_4 x = -1.5$

$4^{-1.5} = x$

$x = \frac{1}{8}$ or 0.125

29 a $\ln x + \ln 3 = \ln 6$

$\ln 3x = \ln 6$

$3x = 6$

$x = 2$

b $e^x + 3e^{-x} = 4$

$e^x + \dfrac{3}{e^x} = 4$

$e^{2x} + 3 = 4e^x$

$e^{2x} - 4e^x + 3 = 0$

let $y = e^x$

$y^2 - 4y + 3 = 0$

$(y-3)(y-3) = 0$

$y = 3$ or 1

$y = e^x$

$e^x = 3$ or $e^x = 1$

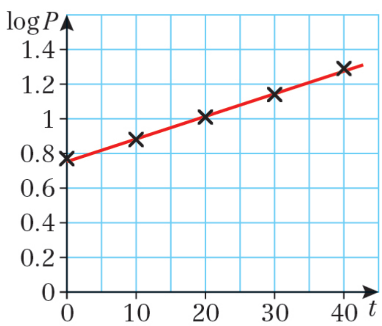
$x = 0$

$x = \ln 3$ or $x = 0$

30 a

Time in years since 1970, t	$\log P$
0	0.77
10	**0.88**
20	**1.01**
30	**1.14**
40	**1.29**

b



30 c As $P = ab^t$

$\log P = \log (ab^t)$

$\log P = \log a + \log b^t$

$\log P = \log a + t \log b$

This is a linear relationship where the gradient is $\log b$ and the intercept is $\log a$.

d Intercept = 0.77

$\log a = 0.77$

$a = 10^{0.77}$

$= 5.888...$

≈ 5.9 (2 s.f.)

$$\text{Gradient} = \frac{1.29 - 0.77}{40 - 0} = \frac{0.52}{40} = 0.013$$

$\log b = 0.013$

$b = 10^{0.013}$

$= 1.03...$

≈ 1.0

$a = 5.9$, $b = 1.0$

31 a

$$\log 2 + \log x = \log y + \log(x + y)$$
$$\log 2x = \log y + \log(x + y)$$
$$\log 2x - \log y = \log(x + y)$$
$$\log \frac{2x}{y} = \log(x + y)$$
$$\frac{2x}{y} = x + y$$
$$2x = xy + y^2$$
$$2x - xy = y^2$$
$$x(2 - y) = y^2$$
$$x = \frac{y^2}{2 - y}$$

b $0 < y < 2$

$y > 0$ given.

$x > 0$ also given, and $y^2 > 0$, so $2 - y$ must be > 0. Hence $y < 2$. Note strict inequality because denominator cannot be 0.

Challenge

1 a 090° means $\sin \theta = 0$

Therefore, $\theta = 0$

b $\cos \theta = 1$

So the vector is 1**i**

Magnitude $= \sqrt{1^2 + 0^2} = 1$

2 a $f'(-3) = k((-3)^2 - 3 - 6) = 0$

$f'(2) = k(2^2 + 2 - 6) = 0$

Using the factor theorem, $x + 3$ and $x - 2$ are factors of $f'(x)$.

So $f'(x) = k(x + 3)(x - 2)$

$= k(x^2 + x - 6)$

As $f(x)$ is cubic, there are no other factors of $f'(x)$.

b $$\int k(x^2 + x - 6)\,dx = \int (kx^2 + kx - 6k)\,dx$$
$$= \frac{kx^3}{3} + \frac{kx^2}{2} - 6kx + c$$

At $(-3, 76)$

$$\frac{k(-3)^3}{3} + \frac{k(-3)^2}{2} - 6k(-3) + c = 76$$
$$-9k + \frac{9k}{2} + 18k + c = 76$$
$$\frac{27k}{2} + c = 76$$

At $(2, -49)$

$$\frac{k(2)^3}{3} + \frac{k(2)^2}{2} - 6k(2) + c = -49$$
$$\frac{8k}{3} + 2k - 12k + c = -49$$
$$-\frac{22k}{3} + c = -49$$

Solving $\frac{27k}{2} + c = 76$ and $-\frac{22k}{3} + c = -49$ simultaneously

$c = 76 - \frac{27k}{2}$ and $c = \frac{22k}{3} - 49$

So $76 - \frac{27k}{2} = \frac{22k}{3} - 49$

$456 - 81k = 44k - 294$

$125k = 750$

$k = 6$, $c = -5$

$$f(x) = \frac{kx^3}{3} + \frac{kx^2}{2} - 6kx + c$$
$$= \frac{6x^3}{3} + \frac{6x^2}{2} - 6(6)x - 5$$
$$= 2x^3 + 3x^2 - 36x - 5$$

3 $\int_0^9 f(x)\ dx = 24.2$

$\int_0^9 (f(x)+3)\ dx$

$= [f'(x)+3x]_0^9$

$= (f'(9)+3(9))-(f'(0)+3(0))$

$= \int_0^9 f(x)\ dx + 27$

$= 24.2 + 27$

$= 51.2$

4 a $f(0) = 0^3 - k(0) + 1 = 1$

$g(0) = e^{2(0)} = e^0 = 1$

Therefore, $f(0) = g(0) = 1$

$P(0, 1)$

b $f'(x) = 3x^2 - k$

Gradient at $x = 0$

$f'(0) = 3(0)^2 - k = -k$

Gradient of $g(x)$ at $x = 0$ is $\frac{1}{k}$

$g'(x) = 2e^{2x}$

$g'(0) = 2e^{2(0)} = 2e^0 = 2$

$\frac{1}{k} = 2$

$k = \frac{1}{2}$